Ivane Javakhishvili Tbilisi State University

LECTURE NOTES

Algebra, Topology and Analysis: C^* and $A_\propto$ Algebras

Summer School/Conference

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Algebra, Topology and Analysis: C^* and A_∞ Algebras

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Summer School/Conference *"Algebra, Topology and Analysis: C^* and A_∞ Algebras"* was the third event in the summer school series (http://www.mathphd.tsu.ge) within the framework of the International Doctoral Program in Mathematics at Ivane Javakhishvili Tbilisi State University, supported by the Shota Rustaveli National Science Foundation and the Volkswagen Foundation. The event was held in Gonio, Batumi, Georgia from 30 August to 3 September 2021.

The two summer schools before were

- Harmonic Analysis, Martingales & Paraproducts, September, 2-6, 2019, Bazaleti, Georgia;
- Operator Algebras, Spectral Theory & Applications to Topological Insulators, September 17-21, 2018, TSU, Tbilisi.

Gonio summer school/conference mainly addressed graduate students and postdoctoral researchers and covered range of topics from algebra, topology and analysis. Plenary talks focused on C^* and A_∞ algebras.

C^*-algebras are the abstract context for understanding self-adjoint operators in Hilbert spaces and go back to quantum mechanics by von Neumann, Heisenberg, and Schrodinger in the 1920s. For the basics we refer the reader to

- https://math.berkeley.edu/ qchu/Notes/208.pdf;
- https://math.berkeley.edu/∼brent/files/209_notes.pdf.

For A_∞ algebras we refer to

- https://arxiv.org/pdf/math/9910179.pdf.

The following contain lectures by:

Tornike Kadeishvili	A_∞-Algebra Structure in Cohomology and its Applications;
Karen Strung	Smale Spaces and Their C*-Algebras;
Bhishan Jacelon	Concentration of Measure;
Andrey Krutov	Torwads to Noncommutative Dynamical Systems;
George Nadareishvili	Approximations of Kasparov Categories of C^*-Algebras;
Karmen Grizelj	Harish–Chandra Map and Primitive Invariants.

Organizers:

M. Bakuradze, V. Baladze, A. Beridze, T. Bokelavadze, A. Gogatishvili, R. Meyer, O. Purtukhia, R. Surmanidze.

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Tbilisi, January 5, 2023

Harish–Chandra Map and Primitive Invariants

Karmen Grizelj, Pavle Pandžić

Let $\mathfrak{g}$ be a semisimple complex Lie algebra. For every $x \in \mathfrak{g}$ define the adjoint action of x on $\mathfrak{g}$ by $\mathrm{ad}_x(y) = [x, y]$, $y \in \mathfrak{g}$. Note that $\mathrm{ad}_x \in \mathrm{End}(\mathfrak{g})$. The Killing form on $\mathfrak{g}$ is a symmetric bilinear form

$$B(x, y) = \mathrm{tr}(\mathrm{ad}_x \, \mathrm{ad}_y), \quad x, y \in \mathfrak{g}.$$

The adjoint action of $\mathfrak{g}$ can be extended to the Clifford algebra $C(\mathfrak{g})$ of $\mathfrak{g}$, the exterior algebra $\wedge\mathfrak{g}$ of $\mathfrak{g}$, and the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Those extensions will be denoted in the same way as the original action.

Denote by J the space of $\mathfrak{g}$-invariants in $\wedge\mathfrak{g}$, that is

$$J = \big(\wedge \mathfrak{g}\big)^{\mathfrak{g}} = \big\{x \in \wedge\mathfrak{g} : \ \mathrm{ad}_y(x) = 0, \ \forall\, y \in \mathfrak{g}\big\}.$$

Consider the augmentation ideal of J:

$$J^+ = \sum_{k>0} J^{(k)}, \quad J^{(k)} = J \cap \wedge^k \mathfrak{g}.$$

The bilinear form B extends to $\wedge\mathfrak{g}$ by determinant, in particular to J. Define the space of primitive invariants P as the B-orthogonal complement of $J^+ \wedge J^+$ in J^+.

Theorem 1 (Hopf–Koszul–Samelson, see [5])**.** *The dimension of the space P is equal to the rank of $\mathfrak{g}$ and the inclusion $P \hookrightarrow J$ extends to an isomorphism of algebras $\wedge P \to J$.*

Consider a $\mathfrak{g}$-module isomorphism $q : \wedge\mathfrak{g} \to C(\mathfrak{g})$. If Z_i denotes an orthonormal basis of $\mathfrak{g}$ with respect to B, then the map is defined by

$$q(Z_{i_1} \wedge Z_{i_2} \wedge \cdots \wedge Z_{i_k}) = Z_{i_1} Z_{i_2} \cdots Z_{i_k}, \quad q(1) = 1.$$

This map is called the Chevalley map or quantization map.

Let the map $\sigma : \wedge\mathfrak{g} \to \mathfrak{g}$ be defined by $\sigma = (-1)^{\frac{k(k-1)}{2}}$ id on $\wedge^k\mathfrak{g}$. For $p, q \in \wedge\mathfrak{g}$, set

$$B_0(p, q) := B(\sigma(p), q).$$

Theorem 2 ([4])**.** *The space $q(J)$ is a Clifford algebra of P with a bilinear form B_0.*

Define a map $\alpha : U(\mathfrak{g}) \to C(\mathfrak{g})$ in the following way: let b_i be a basis of $\mathfrak{g}$ and d_i its B-dual basis. For $x \in \mathfrak{g}$ set

$$\alpha(x) = \frac{-1}{4} \sum [x, b_i] d_i \in C(\mathfrak{g}).$$

Using the universal property, extend this map to the universal enveloping algebra of $\mathfrak{g}$. Denote the image of α by E.

Fix a system of positive roots in $\mathfrak{g}$ and let ρ be the half of the sum of positive roots in $\mathfrak{g}$. Denote by V_ρ the irreducible representation with the highest weight ρ. The following theorem is known as the ρ-decomposition.

Theorem 3 ([4])**.** *We have $E = \mathrm{End}\, V_\rho$ and $C(\mathfrak{g}) = E \otimes J$.*

For $x \in \mathfrak{g}$, let $\iota_x : \wedge\mathfrak{g} \to \wedge\mathfrak{g}$ denote the contraction by x, that is $\iota_x(y) = B(x, y)$ for $y \in \mathfrak{g} \cong \wedge^1\mathfrak{g}$ and extend it to a derivation of degree -1 on $\wedge\mathfrak{g}$.

Theorem 4 ([4])**.** *We have $q(\iota_x(p)) \in E$, $\forall\, x \in \mathfrak{g}$, $p \in P$.*

Theorem 5 ([4]). *Denote by p_i a basis of P and by q_i its dual basis with respect to B_0. Then, for all $x \in \mathfrak{g}$ we have $x = \sum \iota_x(p_i)q_i$.*

Let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a triangular decomposition of $\mathfrak{g}$ (see [3] for details), so

$$C(\mathfrak{g}) = C(\mathfrak{h}) \oplus \left(\mathfrak{n}^+ C(\mathfrak{g}) + C(\mathfrak{g})\mathfrak{n}^-\right).$$

Define the Harish–Chandra map $\mu : C(\mathfrak{g}) \to C(\mathfrak{h})$ as the projection with respect to this decomposition. Note that the map μ is not an algebra homomorphism.

Theorem 6 ([1]). *The map $\mu : C(\mathfrak{g})^{\mathfrak{h}} \to \mathfrak{h}$ is an algebra isomorphism. Furthermore, it restricts to a linear bijection between P and $\mathfrak{h}$.*

Let G be a reductive Lie group, hence there is a Cartan involution on $\mathfrak{g}$, see, for example, [3, § 1]. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition and let $\mathfrak{h}$ be the fundamental Cartan subalgebra, in other words $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ and $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$. If we denote by G a Lie group such that $\mathfrak{g}$ is its Lie algebra and by K its maximal compact subgroup, then the cohomology of G/K is the same as $\mathfrak{k}$-invariants in $\wedge\mathfrak{p}$.

Let $\mathfrak{g} = \mathfrak{sl}(2n+1, \mathbb{R})$, so $\mathfrak{k} = \mathfrak{so}(2n+1)$. Let e_i be a basis of $\mathfrak{p}$ and f_i its B-dual basis. Then the map $\alpha : U(\mathfrak{k}) \to C(\mathfrak{p})$ is defined in this way: for $x \in \mathfrak{k}$, we have

$$\alpha(x) = \sum [x, e_i] f_i$$

and then extend it to $U(\mathfrak{k})$ using the universal property.

Theorem 7. *[2, 6] The algebra $C(\mathfrak{p})$ admits the ρ-decomposition:*

$$C(\mathfrak{p}) = E \otimes J = \alpha(U(\mathfrak{k})) \otimes C(\mathfrak{p})^{\mathfrak{k}}.$$

Define a Harish–Chandra map μ as the projection from $C(\mathfrak{p})$ to $C(\mathfrak{a})$, with respect to the following direct sum:

$$C(\mathfrak{p}) = C(\mathfrak{a}) \oplus \left((\mathfrak{n}^+ \cap \mathfrak{p})C(\mathfrak{p}) + C(\mathfrak{p})(\mathfrak{n}^- \cap \mathfrak{p})\right).$$

Theorem 8. *The map $\mu : C(\mathfrak{p})^{\mathfrak{k}} \to C(\mathfrak{a})$ is an algebra isomorphism.*

Conjecture 1. *We have $\iota_x(p) \in E$, $\forall x \in \mathfrak{p}$, $p \in P(\mathfrak{p})$.*

The conjecture is verified in some examples. The next conjecture is a corollary of Conjecture 1.

Conjecture 2. *The map μ is an isomorphism of vector spaces $P(\mathfrak{p})$ and $\mathfrak{a}$.*

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Concentration of Measure

Bhishan Jacelon

Abstract

Isoperimetric inequalities imply that, probabilistically, Lipschitz functions on high dimensional geometric structures are approximately constant. This phenomenon is known as 'concentration of measure'. In this brief introduction, I will describe these geometric situations and discuss examples related to dynamical systems, in particular, groups of measure-preserving automorphisms of Lebesgue space, and Anosov diffeomorphisms like Arnold's cat map.

1 Concentration of measure

This section and its sequel constitute a very brief introduction to the subject, based on a Part III course delivered by Prof. D. J. H. Garling at the University of Cambridge in 2007.

Let (X, d, μ) be a *metric measure space*, that is, μ is a probability measure defined on the Borel σ-algebra generated by the open sets of a metric space (X, d). We define the *concentration function* of (X, d, μ) to be

$$\alpha_X(\varepsilon) = \sup \left\{ \mu(A_\varepsilon^c) \mid \ A \subseteq X \ \text{ Borel}, \ \mu(A) \geq \frac{1}{2} \right\}. \tag{1.1}$$

Here, $A_\varepsilon = \{x \in X \mid \ d(x, A) \leq \varepsilon\}$ is the ε-*neighbourhood* of A in X, and A_ε^c denotes its complement $X \setminus A_\varepsilon$. We will consider two examples:

(i) the sphere $S^n \subseteq \mathbb{R}^{n+1}$ equipped with geodesic distance d and Haar measure μ (that is, μ is the unique isometry-invariant Borel probability measure on S^n);

(ii) the symmetric group S_n equipped with the uniform measure $\mu(A) = \frac{|A|}{n!}$ and *Hamming distance* $d(\sigma, \tau) = \frac{1}{n} |\{i \mid \ \sigma(i) \neq \tau(i)\}|$.

We will see that these both form *normal Lévy families*: there are constants $c_1, c_2 > 0$ such that, for $X_n = S^n$ or $X_n = S_n$, the *concentration inequality*

$$\alpha_{X_n}(\varepsilon) \leq c_1 \exp(-c_2 n \varepsilon^2) \tag{1.2}$$

holds for all $n \in \mathbb{N}$. Our motivating slogan is that on high dimensional spaces X_n with concentration (that is, when we have a concentration inequality available and n is large), Lipschitz functions defined on X_n are approximately constant (that is, close to their medians with high probability). Let us see why this is.

Suppose that $f : X = X_n \to \mathbb{R}$ is 1-*Lipschitz*, that is,

$$|f(x) - f(y)| \leq d(x, y) \quad \forall\, x, y \in X, \tag{1.3}$$

and that M_f is a *median* for f, that is,

$$\mu\big(\{x \in X \mid \ f(x) \leq M_f\}\big) \geq \frac{1}{2} \ \text{ and } \ \mu\big(\{x \in X \mid \ f(x) \geq M_f\}\big) \geq \frac{1}{2},$$

which we can write concisely as

$$\mu(f \leq M_f) \geq \frac{1}{2} \ \text{ and } \ \mu(f \geq M_f) \geq \frac{1}{2}. \tag{1.4}$$

Let $\varepsilon > 0$. Then, by (1.1), (1.3) and (1.4),

$$\mu(f > M_f + \varepsilon) \leq \alpha_X(\varepsilon) \text{ and } \mu(f < M_f - \varepsilon) \leq \alpha_X(\varepsilon),$$

so by (1.2),

$$\mu\big(|f - M_f| > \varepsilon\big) \leq 2c_1 \exp(-c_2 n \varepsilon^2). \tag{1.5}$$

This phenomenon is indeed the reason for the definition (1.1) of the function α_X.

Concentration of measure on S^n will follow from the *spherical isoperimetric inequality*, based on the classical geometric problem which we formulate here as minimising the perimeter $\liminf\limits_{\varepsilon \to 0} \frac{1}{\varepsilon}(\mu(A_\varepsilon) - \mu(A))$ of a region A of given area $\mu(A)$. On the other hand, concentration on S_n will follow from a *martingale inequality*, and we will see as a consequence the *extreme amenability* of the group of measure-preserving automorphisms of Lebesgue space (also the unitary groups of certain operator algebras).

2 Spherical isoperimetry

Recall that we work with geodesic distance d and Haar measure μ on the sphere $S^n \subseteq \mathbb{R}^{n+1}$. A spherical *cap* is a set of the form $C = B_r(n) = \{x \in S^n \mid d(x, n) \leq r\}$, where $n = (0, \dots, 0, 1)$ is the north pole.

Theorem 2.1 (Spherical isoperimetry)**.** *If $A \subseteq S^n$ is a Borel set and $C \subseteq S^n$ is a cap of equal measure, then $\mu(A_\varepsilon) \geq \mu(C_\varepsilon)$ for every $\varepsilon > 0$.*

Proof. Approximating by a sufficiently fine net, we may assume that A is closed. We will use the fact that the set K of nonempty closed subsets of S^n is compact when equipped with the Hausdorff metric

$$\rho(X, Y) = \inf\big\{r > 0 \mid X \subseteq Y_r,\ Y \subseteq X_r\big\}.$$

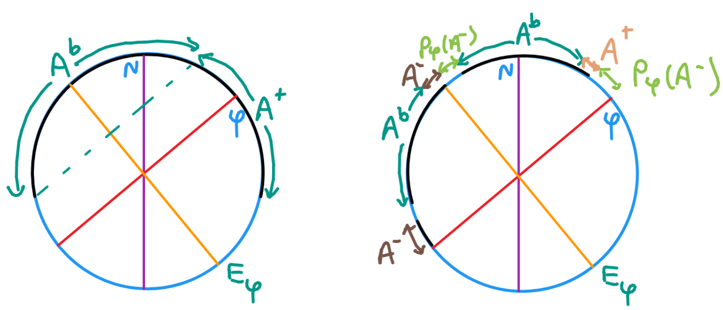
We denote by $\langle \cdot\,, \cdot \rangle$ the usual dot product on $\mathbb{R}^{n+1}$. Given $\varphi \in S^n$ with $\langle \varphi, n \rangle > 0$, let

$$\begin{aligned} E_\varphi &= \big\{x \in S^n \mid \langle \varphi, x \rangle = 0\big\} \quad \text{(the } \varphi\text{-equator)}, \\ K_\varphi^+ &= \big\{x \in S^n \mid \langle \varphi, x \rangle \geq 0\big\} \quad \text{(northern hemisphere)}, \\ K_\varphi^- &= \big\{x \in S^n \mid \langle \varphi, x \rangle < 0\big\} \quad \text{(southern hemisphere)} \end{aligned}$$

and let P_φ be reflection in E_φ. Explicitly, $P_\varphi(x) = x - 2\langle \varphi, x \rangle \varphi$.

The proof is by *symmetrisation*: a procedure of rearranging a set by isometries (specifically, reflections) to increase the measure of its intersection with the cap C. With this in mind, and given φ as above and a closed subset $A \subseteq S^n$, let

$$\begin{aligned} A_\varphi^b &= \big\{x \in A \mid P_\varphi(x) \in A\big\}, \\ A_\varphi^+ &= \big\{x \in A \cap K_\varphi^+ \mid P_\varphi(x) \notin A\big\}, \\ A_\varphi^- &= \big\{x \in A \cap K_\varphi^- \mid P_\varphi(x) \notin A\big\}, \\ A_\varphi^* &= A_\varphi^b \cup A_\varphi^+ \cup P_\varphi(A_\varphi^-). \end{aligned}$$



It is straightforward to check that A^*_φ is a closed subset such that

$$\mu(A^*_\varphi) = \mu(A) \text{ and } (A^*_\varphi)_\varepsilon \subseteq (A_\varepsilon)^*_\varphi,$$

so in particular,

$$\mu((A^*_\varphi)_\varepsilon) \leq \mu(A_\varepsilon).$$

It is also easy to show that

$$X := \Big\{ B \in K \mid \ \mu(B) = \mu(A) \text{ and } \mu(B_\varepsilon) \leq \mu(A_\varepsilon) \ \forall\, \varepsilon > 0 \Big\}$$

is a closed subset of K such that

$$B \in X \implies B^*_\varphi \in X \ \ \forall\, \varphi \in S^n \ \text{ with } \ \langle \varphi, n \rangle > 0.$$

Let Y be the smallest closed subset of K containing A that has this property, that is, $B^*_\varphi \in Y$ whenever $B \in Y$ and $\varphi \in S^n$ with $\langle \varphi, n \rangle > 0$. Then $Y \subseteq X$, so for every $B \in Y$ and $\varepsilon > 0$, $\mu(B) = \mu(A)$ and $\mu(B_\varepsilon) \leq \mu(A_\varepsilon)$.

Since K is compact, so is Y, so there exists $B_0 \in Y$ such that $\mu(B \cap C)$ attains its maximum on Y at $B = B_0$. The theorem will be proved once we have shown that $C \subseteq B_0$.

Suppose that $C \nsubseteq B_0$. Then, there exist $x \in S^n$ and $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq C \setminus B_0$. Hence, $\mu(C \setminus B_0) > 0$, so $\mu(B_0 \setminus C) > 0$ too (otherwise, $\mu(B_0) = \mu(B_0 \cap C) < \mu(C) = \mu(A) = \mu(B_0)$). Cover $B_0 \setminus C$ by finitely many balls of radius $\frac{\varepsilon}{3}$ and choose one, say $B_{\frac{\varepsilon}{3}}(y)$, such that $\mu(B_{\frac{\varepsilon}{3}}(y) \cap B_0 \setminus C) > 0$. Note that $d(x,y) \geq \frac{2\varepsilon}{3}$ (otherwise, $B_\varepsilon(x) \cap B_0 \neq d\mu$). Let $\varphi = \frac{x-y}{\|x-y\|}$, so that $P_\varphi(y) = x$ and so $\langle \varphi, n \rangle > 0$. Then, $P_\varphi(B_{\frac{\varepsilon}{3}}(y) \cap B_0 \setminus C) \subseteq C$, so $\mu((B_0)^*_\varphi \cap C) > \mu(B_0 \cap C)$, which contradicts the maximality of $\mu(B_0 \cap C)$. □

Using integration to bound the measure of a spherical cap, we obtain the following.

Corollary 2.2. *The concentration function on S^n satisfies*

$$\alpha_{S^n}(\varepsilon) \leq \sqrt{\frac{\pi}{8}} \exp\Big(- \frac{n-1}{2}\, \varepsilon^2 \Big).$$

We conclude this section with some other examples of isoperimetric inequalities.

(i) A consequence of the Brunn–Minkowski inequality (see for example [1]) is that in $\mathbb{R}^n$, equipped with Euclidean distance and Lebesgue measure λ, the isoperimetric problem is solved by the ball. Equipped instead with *Gaussian measure*

$$\gamma_n(A) = (2\pi)^{-\frac{n}{2}} \int_A \exp\Big(- \frac{\|x\|^2}{2} \Big)\, d\lambda(x),$$

the problem is solved by half-spaces $H_s = \{x \in \mathbb{R}^n \mid \ x_n \leq s\}$, yielding the *Gaussian isoperimetric inequality*

$$\alpha_{(\mathbb{R}^n, \gamma_n)}(\varepsilon) \leq \frac{1}{\varepsilon\sqrt{2\pi}} \exp\Big(- \frac{\varepsilon^2}{2} \Big).$$

(ii) On the hypercube $Q_n = \{0,1\}^n$, equipped with uniform measure and Hamming distance $\|x - y\|_1 = \{i \mid \ x_i \neq y_i\}$, the problem is solved by initial segments in the reverse lexicographic order. This is Harper's Theorem [10]. Therefore,

$$\alpha_{Q_n}(\varepsilon) \leq \exp\Big(- \frac{2}{n}\, \varepsilon^2 \Big). \tag{2.1}$$

To see (2.1), first note that an initial segment I of measure $\geq \frac{1}{2}$ must contain an element $x \in Q_n$ of length $l(x) := \|x\|_1 \geq \frac{n}{2}$ if n is even, or the largest x of length $\frac{n-1}{2}$ if n is odd (otherwise, $|I| < 2^{n-1} = \frac{1}{2}|Q_n|$). Then, since I is an initial segment, $B_{\lceil \frac{n-1}{2} \rceil}(0) \subseteq I$. So, for any $\varepsilon > 0$,

$$I^c_\varepsilon \subseteq \Big\{ x \in Q_n \mid \ l(x) > \frac{n-1}{2} + \varepsilon \Big\}.$$

By isoperimetry,

$$\begin{aligned}
\alpha_{Q_n}(\varepsilon) &= \mu(I_\varepsilon^c)\\
&\le \mu\Big(l > \frac{n-1}{2} + \varepsilon\Big)\\
&= \text{probability of} > \frac{n-1}{2} + \varepsilon \text{ heads when a fair coin is tossed } n \text{ times}\\
&= \mathbb{P}(s_n > 2\varepsilon + 1)\\
&\le \mathbb{P}(s_n \ge 2\varepsilon),
\end{aligned}$$

where $s_n = \sum_{i=1}^{n} \varepsilon_i$ is the sum of n independent identically distributed Bernoulli random variables with $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}$. Each ε_i is *sub-Gaussian* in the sense of Kahane [11]:

$$\mathbb{E}(e^{t\varepsilon_i}) = \cosh(t) \le e^{\frac{1}{2}t^2},$$

hence so is s_n: by independence,

$$\mathbb{E}(e^{ts_n}) \le e^{\frac{n}{2}t^2}.$$

It follows from Markov's inequality (see for example [16, Ch. 6]) with $t = \frac{2\varepsilon}{n}$ that

$$\mathbb{P}(s_n \ge 2\varepsilon) \le e^{-2\varepsilon t}\mathbb{E}(e^{ts_n}) \le e^{-\frac{2}{n}\varepsilon^2}.$$

3 The symmetric group

The content of this section appears in [13, Ch. 7].

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$ and $f \in L^1(\Omega, \mathcal{F}, \mu)$. Then,

$$\nu(A) := \int_A f\, d\mu, \quad A \in \mathcal{G}$$

defines a measure on $\mathcal{G}$ that is absolutely continuous with respect to μ. There is therefore a unique $h \in L^1(\Omega, \mathcal{G}, \mu)$ such that

$$\int_A f\, d\mu = \int_A h\, d\mu \;\; \forall\, A \in \mathcal{G}$$

(that is, h is the Radon–Nikodym derivative $\frac{d\nu}{d\mu}$). We call h the *conditional expectation of f with respect to $\mathcal{G}$*, written $h = \mathbb{E}(f \mid \mathcal{G})$.

The operator $f \mapsto \mathbb{E}(f \mid \mathcal{G})$ is a positive linear map of norm one on all L^p spaces ($1 \le p \le \infty$) such that:

(i) $\mathbb{E}(\mathbb{E}(f \mid \mathcal{G}) \mid \mathcal{G}') = \mathbb{E}(f \mid \mathcal{G}')$ for every $\mathcal{G}' \subseteq \mathcal{G}$;

(ii) $\mathbb{E}(g \cdot f \mid \mathcal{G}) = g \cdot \mathbb{E}(f \mid \mathcal{G})$ for every $g \in L^\infty(\Omega, \mathcal{G}, \mu)$;

(iii) $\mathbb{E}(f \mid \mathcal{G}) = \mathbb{E}f = \int_\Omega f\, d\mu$ if $\mathcal{G} = \{\varnothing, \Omega\}$.

A *martingale* with respect to a sequence of σ-algebras $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}$ is a sequence $f_1, f_2, \ldots$ in $\in L^1(\Omega, \mathcal{F}, \mu)$ such that for every i, $\mathbb{E}(f_{i+1} \mid \mathcal{F}_i) = f_i$.

A special case of interest to us here is when Ω is a finite set, μ is the uniform measure on Ω, $(\Omega_i)_{i=1}^k$ is a sequence of partitions of Ω such that for every i, Ω_{i+1} refines Ω_i, and $\mathcal{F}_i$ is defined to be the σ-algebra generated by Ω_i. In this case, the conditional expectation $\mathbb{E}(f \mid \mathcal{F}_i)$ of a function f on Ω is the function that is constant on *atoms of $\mathcal{F}_i$* (that is, $\mathcal{F}_i$-measurable sets of minimal positive measure), the constant being the average of the values of f on the atom.

Using the basic properties of the conditional expectation enumerated above, together with the inequality $e^x \le x + e^{x^2}$, it is not difficult to prove the following.

Lemma 3.1. *Let $f \in L^\infty(\Omega, \mathcal{F}, \mu)$ and*

$$\{\varnothing, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{F}$$

be an increasing sequence of σ-algebras. For $1 \le j \le n$, write $d_j = \mathbb{E}(f \mid \mathcal{F}_j) - \mathbb{E}(f \mid \mathcal{F}_{j-1})$. Then for every $c > 0$,

$$\mu\big(|f - \mathbb{E}f| \ge c\big) \le 2\exp\Big(-\frac{c^2}{4\sum \|d_j\|_\infty^2}\Big).$$

Theorem 3.2 (Maurey [12])**.** *The symmetric groups S_n, with uniform measure and Hamming distance, form a normal Lévy family with constants $c_1 = 2$ and $c_2 = \frac{1}{64}$, that is, for every $\varepsilon > 0$ and every $A \subseteq S_n$ of size $\ge \frac{n!}{2}$,*

$$\mu(A_\varepsilon^c) \le 2\exp\Big(-\frac{n}{64}\,\varepsilon^2\Big).$$

Proof. Here is a sketch of the argument. For each $1 \le j \le n$, let Ω_j be the partition

$$\Omega_j = \big\{A_{i_1,\dots,i_j} \mid 1 \le i_1, \dots, i_j \le n \text{ distinct}\big\},$$

where $A_{i_1,\dots,i_j} = \{\pi \in S_n \mid \pi(1) = i_1, \dots, \pi(j) = i_j\}$, and let $\mathcal{F}_j = \sigma(\Omega_j)$, so that

$$\{\varnothing, S_n\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = 2^{S_n}.$$

The key fact about this sequence of partitions is the following. For any atom $A_{i_1,\dots,i_j} \in \mathcal{F}_j$ and any two atoms $B = A_{i_1,\dots,i_j,r}, C = A_{i_1,\dots,i_j,s} \in \mathcal{F}_{j+1}$ contained in A, there exists a bijection $\varphi : B \to C$ such that $d(b, \varphi(b)) \le \frac{2}{n}$ for every $b \in B$ (namely, $\varphi(\pi) = (r\ s) \circ \pi$).

From this, one can deduce that for any 1-Lipschitz function f on S_n, the martingale $f_j = \mathbb{E}(f \mid \mathcal{F}_j)$ satisfies $\|d_j\|_\infty \le \frac{2}{n}$ for $1 \le j \le n$. Then, by Lemma 3.1, for any such f and any $c > 0$,

$$\mu\big(|f - \mathbb{E}f| \ge c\big) \le 2\exp\Big(-\frac{n}{16}\,c^2\Big). \tag{3.1}$$

This in particular applies to the distance $f = d(\,\cdot\,, A)$ from any $A \subseteq S_n$, and the constant $c = 4(\frac{1}{n}\log 4)^{\frac{1}{2}}$, to give

$$\mu\Big(\big|d(\,\cdot\,, A) - \mathbb{E}d(\,\cdot\,, A)\big| < 4\Big(\frac{1}{n}\log 4\Big)^{\frac{1}{2}}\Big) > \frac{1}{2}\,.$$

Now fix $A \subseteq S_n$ with $\mu(A) \ge \frac{1}{2}$. Then $\mu(d(\,\cdot\,, A) = 0) \ge \frac{1}{2}$, so there exists $\pi \in S_n$ such that $d(\pi, A) = 0$ and

$$\big|d(\pi, A) - \mathbb{E}d(\,\cdot\,, A)\big| < 4\Big(\frac{1}{n}\log 4\Big)^{\frac{1}{2}}.$$

In other words, $\mathbb{E}d(\,\cdot\,, A) < 4(\frac{1}{n}\log 4)^{\frac{1}{2}}$. It then follows from (3.1) that for any $c > 0$,

$$\mu\Big(d(\,\cdot\,, A) \ge c + 4\Big(\frac{1}{n}\log 4\Big)^{\frac{1}{2}}\Big) \le 2\exp\Big(-\frac{n}{16}\,c^2\Big).$$

For any $\varepsilon > 8(\frac{1}{n}\log 4)^{\frac{1}{2}}$, taking $c = \frac{\varepsilon}{2} > 4(\frac{1}{n}\log 4)^{\frac{1}{2}}$ then gives

$$\mu(A_\varepsilon^c) = \mu\big(d(\,\cdot\,, A) > \varepsilon\big) \le 2\exp\Big(-\frac{n}{64}\,\varepsilon^2\Big).$$

For $\varepsilon \le 8(\frac{1}{n}\log 4)^{\frac{1}{2}}$,

$$\mu(A_\varepsilon^c) \le \mu(A^c) \le \frac{1}{2} \le 2\exp\Big(-\frac{n}{64}\,\varepsilon^2\Big),$$

so the inequality holds for all $\varepsilon > 0$. ∎

4 Extreme amenability

A topological group G is *amenable* if every continuous affine action of G on a nonempty compact convex set has a fixed point, or equivalently if there is a left-invariant mean on the space $C_r^b(G)$ of bounded right uniformly continuous functions on G.

Here, 'right uniformly continuous' means uniformly continuous with respect to the *right uniformity on* G, that is, the coarsest uniformity on G compatible with its underlying topology such that each right-multiplication map $g \mapsto gh$ is uniformly continuous. A *left-invariant mean* is a function $m : C_r^b(G) \to \mathbb{C}$ that is positive, linear, unital and invariant under the left action of G on $C_r^b(G)$, that is, $m({}^g f) = m(f)$ for every $f \in C_r^b(G)$ and $g \in G$, where ${}^g f(x) = f(g^{-1}x)$.

If G is locally compact, amenability is also equivalent to the existence of an invariant mean on the larger C*-algebra $L^\infty(G)$ of measurable functions $G \to \mathbb{C}$ that are essentially bounded with respect to Haar measure. There are many, many more equivalent conditions; an excellent reference is [14].

Examples of amenable groups include compact groups, abelian groups, and the unitary groups of injective von Neumann algebras equipped with the ultraweak topology (see [6]), but *not* groups containing the free group $\mathbb{F}_2$ as a closed subgroup (which admit paradoxical decompositions, as in the Banach–Tarski paradox).

Relaxing the 'affine' constraint leads to a much stronger notion of amenability: G is *extremely amenable* if *every* continuous action of G on a nonempty compact space has a fixed point, or equivalently if $C_r^b(G)$ has a *multiplicative* invariant mean.

Extremely amenable groups must be large.

Theorem 4.1 (Veech [15]). *Any locally compact group admits a free action on some compact space, so is* not *extremely amenable.*

On the other hand, Gromov and Milman used concentration of measure to prove the following.

Theorem 4.2 (Gromov–Milman [8]). *The unitary group of infinite-dimensional Hilbert space is extremely amenable under the strong operator topology.*

Here, the *strong operator topology* (SOT) on the space of bounded linear operators on the Hilbert space H is defined by $T_j \to T$ iff $\|T_j x - Tx\| \to 0$ for every $x \in H$. On the space $\mathcal{U}(H)$ of unitary operators on H, this coincides with the *weak operator topology* (WOT), defined by $T_j \to T$ iff $\langle T_j x, T_j y\rangle \to \langle Tx, Ty\rangle$ for every $x, y \in H$.

To illustrate the idea, let us examine the proof of the following related result of Giordano and Pestov.

Theorem 4.3 (Giordano–Pestov [7]). *The group* $\mathrm{Aut}\,(\mathrm{X}, \mu)_{\mathrm{w}}$ *of all measure-preserving automorphisms of a standard nonatomic (σ-) finite measure space is extremely amenable under the weak topology.*

In the finite case, we may assume up to isomorphism that (X, μ) is $[0, 1]$ with Lebesgue measure λ. $\mathrm{Aut}\,(\mathrm{X}, \mu)$ is then the group of (equivalence classes of, that is, up to sets of measure zero) invertible maps $T : [0, 1] \to [0, 1]$ such that $\lambda(T^{-1}E) = \lambda(E)$ for all measurable E.

Each such T induces a unitary operator

$$U_T : L^2([0, 1], \lambda) \to L^2([0, 1], \lambda), \quad f \mapsto f \circ T.$$

The *weak topology* on $\mathrm{Aut}\,([0, 1], \lambda)$ is the restriction of the strong (equivalently, weak) operator topology on the image of the map $T \mapsto U_T$. Equivalently, $T_j \to T$ iff $\lambda(T_j E \bigtriangleup TE) \to 0$ for every measurable set E.

The *uniform topology* on $\mathrm{Aut}\,([0, 1], \lambda)$ is induced by the left-invariant metric

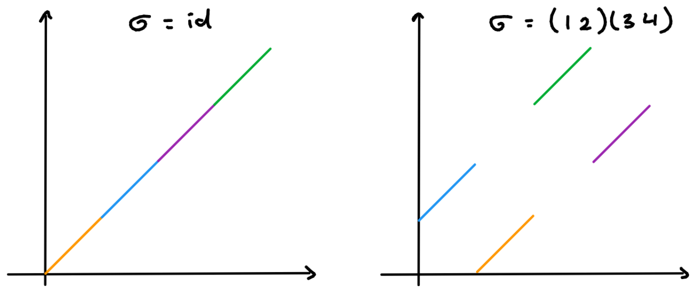
$$d(\sigma, \tau) = \lambda\Big(\big\{x \in [0, 1] \mid \sigma(x) \neq \tau(x)\big\}\Big)$$

and is strictly finer than the weak topology.

For each $n \in \mathbb{N}$, the symmetric group S_{2^n} embeds into $\mathrm{Aut}\,([0,1],\lambda)$ via *interval exchange transformations* of the dyadic intervals

$$\Big(\frac{k-1}{2^n}, \frac{k}{2^n}\Big), \quad 1 \le k \le 2^n.$$

Under these embeddings, the uniform metric restricts to Hamming distance.



The proof of the following can be found in [9].

Theorem 4.4 (Weak Approximation Theorem). *The increasing union* $\bigcup_{n\in\mathbb{N}} S_{2^n}$ *is weakly dense in* $\mathrm{Aut}\,([0,1],\lambda)$.

To prove Theorem 4.3, it therefore suffices to prove that $G = \bigcup_{n\in\mathbb{N}} S_{2^n}$ is extremely amenable.

Denote Haar measure on S_{2^n} by μ_n. The space $C_r^b(G)$ is a commutative unital C*-algebra (whose spectrum is the *Samuel compactification of* G), so its state space is weak*-compact. We may therefore assume that the states

$$\varphi_n : C_r^b(G) \to \mathbb{C}, \quad \varphi_n(f) = \int\limits_G f\, d\mu_n$$

converge weak* to a state φ. We will show that φ is multiplicative and G-invariant.

For every $f \in C_r^b(G)$ and $n \in \mathbb{N}$, let $M_n(f)$ be a μ_n-median of f (as in § 1) and let $L_f > 0$ be such that f varies by at most ε on entourages of width at most $L_f\varepsilon$. Then, by Theorem 3.2 and the same argument that we used to obtain (1.5),

$$\mu\big(|f - M_n(f)| > \varepsilon\big) \le 2\exp\Big(-\frac{2^n}{64}\, L_f^2\varepsilon^2\Big) \;\; \forall\, \varepsilon > 0.$$

This implies that

$$\int\limits_G |f - M_n(f)|\, d\mu_n \to 0 \;\; \forall\, f \in C_r^b(G)$$

and therefore, for every $f, g \in C_r^b(G)$,

$$\begin{aligned}\Big|\int fg - \int f \int g\Big| \le \int |f - M_n(f)|\,|g| + |M_n(f)| \int |g - M_n(g)| \\ + |M_n(g)| \int |M_n(f) - f| + \int |f| \int |M_n(g) - g| \\ \to 0 \;\text{ as }\; n \to \infty\end{aligned}$$

(where each integral is with respect to μ_n), so $\varphi(fg) = \varphi(f)\varphi(g)$.

Left invariance holds because each μ_n is invariant: given $f \in C_r^b(G)$ and $g \in S_{2^m} \subseteq \bigcup_{n\ge m} S_{2^n}$,

$$\varphi({}^g f) = \lim_{n\to\infty} \int {}^g f\, d\mu_n = \lim_{n\to\infty} \int f\, d\mu_n = \varphi(f).$$

This completes the proof.

Remark 4.5.

(i) *(Gromov–Milman)* More generally, if G is a *Lévy group*, that is, $G = \overline{\bigcup_{i\in I} K_i}$, where the K_i are subgroups of G whose Haar measures concentrate with respect to the right uniformity, then G is extremely amenable.

(ii) *(Giordano–Pestov)* With the *uniform* topology, $\mathrm{Aut}\,([0,1],\lambda)$ is *not* amenable.

5 Anosov diffeomorphisms

By a *dynamical system*, we will mean a probability measure preserving transformation $f : X \to X$ of a probability space (X, Σ, μ). Measurements of the system are made via *observables*: functions φ from the *state space* X to, say, $\mathbb{R}$ that may be required to be, for example, measurable or continuous or Lipschitz.

Given an observable φ, its *spatial average* is $\int_X \varphi\, d\mu$, and its *finite time average* over $n-1$ iterations of the system, with initial state $x \in X$, is

$$A_n\varphi(x) = \frac{1}{n}\big(\varphi(x) + \varphi(fx) + \cdots + \varphi(f^{n-1}x)\big).$$

If μ is *ergodic* with respect to f, that is,

$$E \in \Sigma,\ \ \mu(f^{-1}E \mathbin{\triangle} E) = 0 \implies \mu(E) = 0 \ \text{ or } \ \mu(E) = 1$$

(or equivalently, if every f-invariant (up to measure zero) measurable function $X \to \mathbb{R}$ is almost surely constant), and φ is a measurable observable, then by *Birkhoff's Ergodic Theorem* (see [2, Theorem 4.5.5]),

$$\lim_{n\to\infty} A_n\varphi(x) = \int_X \varphi\, d\mu \ \text{ almost surely.}$$

In this sense, the finite time averages can be thought of as statistical estimators of the spatial average.

Remark 5.1.

(i) If X is compact, then there exists $G \in \Sigma$ of full measure such that for every continuous φ and $x \in G$, $\lim_{n\to\infty} A_n\varphi(x) = \int_X \varphi\, d\mu$. In other words, there is a common set of *generic points* for all continuous observables.

(ii) If μ is not ergodic, then the finite time averages converge to the conditional expectation of φ with respect to the σ-algebra of f-invariant sets.

Question 5.2. How fast do the finite time averages converge to the spatial average?

Not much can be said without imposing some regularity on f, as outlined for example below. Otherwise, convergence for $x \in G$ can be arbitrarily slow.

Definition 5.3. Let M be a compact Riemannian manifold. A diffeomorphism $f : M \to M$, whose derivative we denote by Df, is called an *Anosov diffeomorphism* if the tangent space splits into Df-invariant sub-bundles, $TM = E^s \oplus E^u$, such that Df is *uniformly expanding* on E^u and *uniformly contracting* on E^s.

This means the following:

(i) for every $x \in M$,

$$(Df)_x(E^s(x)) = E^s(f(x)) \ \text{ and } \ (Df)_x(E^u(x)) = E^u(f(x));$$

(ii) there are constants $C > 0$ and $0 < \lambda < 1$ such that, for every $x \in M$ and $n \geq 0$,

$$\|D(f^n)_x \xi\| \leq C\lambda^n \|\xi\| \ \ \forall \xi \in E^s(x),$$
$$\|D(f^{-n})_x \eta\| \leq C\lambda^{-n} \|\eta\| \ \ \forall \eta \in E^u(x).$$

Here, $D(f^n)_x$ denotes the derivative of $\overbrace{f \circ \cdots \circ f}^{n}$ at x.

Given an Anosov diffeomorphism $f : M \to M$ and $x \in M$, the set

$$W^s(x) := \left\{ y \in M \mid \lim_{n\to\infty} d(f^n x, f^n y) = 0 \right\}$$

(the convergence being necessarily exponentially fast) is an immersed submanifold of M, called the *stable manifold* at x. The *stable subspace* $E^s(x)$ is tangent at x to $W^s(x)$. The *unstable manifold* $W^u(x)$ is defined similarly.

Example 5.4 (Arnold's cat map)**.** Let M be the torus $\mathbb{R}^2/\mathbb{Z}^2$ and define

$$f : M \to M, \quad f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \mod 1.$$

This is an example of a *hyperbolic toral automorphism*. The matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ has eigenvalues $\lambda = \frac{1}{2}(3+\sqrt{5}) > 1$ and $\frac{1}{\lambda} < 1$. At $v = \begin{pmatrix} x \\ y \end{pmatrix}$, the derivative $(Df)_v$ has matrix A; it expands by a factor of λ along the λ-eigenvector $v_u = \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) \\ 1 \end{pmatrix}$ and contracts by λ along the orthogonal $\frac{1}{\lambda}$-eigenvector $v_s = \begin{pmatrix} \frac{1}{2}(1-\sqrt{5}) \\ 1 \end{pmatrix}$ (note the irrational slopes). The unstable subspace $E^u(v)$ is tangent to the geodesic at v that is parallel to v_u, so $W^u(v)$ winds densely over the torus, as does $W^s(v)$.

The reason that f is called 'Arnold's cat map' (after Vladimir Arnold) is that points with rational coordinates have periodic orbits, so a picture of a cat, though for a time distorted, will eventually return to normal after finitely many iterations.

The following two results appear in [2] as Theorem 5.10.3 and Theorem 6.3.1.

Theorem 5.5. *The following are equivalent for an Anosov map $f : M \to M$:*

(i) *every $x \in M$ is* non-wandering*: for every open set $U \subseteq M$ containing x, there exists $n \in \mathbb{N}$ such that $f^n U \cap U \neq \varnothing$;*

(ii) *f is* irreducible*: for every nonempty open sets $U, V \subseteq sM$, there exists $n \in \mathbb{N}$ such that $f^n U \cap V \neq \varnothing$;*

(iii) *f is* mixing*: for every nonempty open sets $U, V \subseteq M$, there exists $n \in \mathbb{N}$ such that for every $m \geq n$, $f^m U \cap V \neq \varnothing$;*

(iv) *every stable manifold $W^s(x)$ is dense in M;*

(v) *every unstable manifold $W^u(x)$ is dense in M.*

This in particular implies that the set of periodic points of an irreducible Anosov system are dense, and that such a system is an example of a *Smale space*.

Theorem 5.6 (Anosov)**.** *A C^2 Anosov diffeomorphism that preserves a smooth measure $\mu(A) = \int_A q(x)\, dm(x)$ (where $q : M \to \mathbb{R}$ is continuous and m is the Riemannian volume form on M) is ergodic.*

In the situation of Theorem 5.6, recall that Birkhoff's Ergodic Theorem tells us that, with μ the volume measure, the set

$$\Big\{x \in M \mid \lim_{n\to\infty} A_n\varphi(x) = \int_M \varphi\, d\mu \text{ for every continuous } \varphi\Big\}$$

has full volume. Even if f does not preserve volume in the sense of Theorem 5.6, as long as f is irreducible and C^2, there exists a unique f-invariant measure μ, called the *Sinai–Ruelle–Bowen* (SRB) measure of f, for which this still holds (see [18]).

The following concentration of measure inequality for the SRB measure of such an Anosov system (or more generally, a system modelled by a *Young tower* with exponential tails, as in [17]) was proved using martingale techniques (see [4, 5]).

Theorem 5.7. *There is a constant $C > 0$ such that for every $n \in \mathbb{N}$ and every function $K(x_0, x_1, \dots, x_{n-1})$ that is 1-Lipschitz in each coordinate,*

$$\int_M \exp\Big(K(x, fx, \dots, f^{n-1}x) - \int_M K(y, fy, \dots, f^{n-1}y)\, d\mu(y)\Big)\, d\mu(x) \le \exp(Cn).$$

Corollary 5.8. *For every $\varepsilon > 0$ and every 1-Lipschitz observable φ,*

$$\mu\bigg(\Big\{x \in M \mid \Big|A_n\varphi(x) - \int_M \varphi\, d\mu\Big| > \varepsilon\Big\}\bigg) \le 2\exp\Big(-\frac{n}{4C}\,\varepsilon^2\Big).$$

Proof. Take $K(x_0, \dots, x_{n-1}) = \varphi(x_0) + \dots + \varphi(x_{n-1})$ and use the *Chernoff bounding trick*, that is, an application of Markov's inequality similar to our use in § 2 in the context of sub-Gaussian random variables. See [3, § 5] for a fuller exposition. □

This allows to conclude with an answer to Question 5.2: for C^2 Anosov systems, the finite time averages of a Lipschitz observable converge in SRB measure to its spatial average exponentially fast.

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A_∞-Algebra Structure in Cohomology and its Applications

Tornike Kadeishvili

The main method of algebraic topology is to assign to a topological a space certain algebraic object (model) and to study this relatively simple algebraic object instead of complex geometric one.

Examples of such models are chain and cochain complexes, homology and homotopy groups, cohomology algebras, etc.

The main problem here is to find models that classify spaces up to some equivalence relation, such as homeomorphism, homotopy equivalence, rational homotopy equivalence, etc.

Usually such models are not *complete*: the equivalence of models does not guarantee the equivalence of spaces. They can just distinguish spaces.

The models which carry richer algebraic structure contain more information about the space. For example the model "cohomology algebra" allows to distinguish spaces which cannot be distinguished by the model "cohomology groups".

Here we are going to present one more additional algebraic structure on cohomology, which was constructed in [15, 16], namely, we show that on cohomology $H^*(X,R)$ there exists Stasheff's A_∞-algebra structure. This structure consists of a collection of operations

$$\big\{m_i : H^*(X,R) \otimes \cdots (i \text{ times}) \cdots \otimes H^*(X,R) \to H^*(X,R),\ i = 2,3,\dots\big\}.$$

In fact this structure extends the usual structure of the cohomology algebra: the first operation $m_2 : H^*(X,R) \otimes H^*(X,R) \to H^*(X,R)$ coincides with the cohomology multiplication.

Stasheff's A_∞ algebra is a sort of Strong Homotopy Associative Algebra, the operation m_3 is a homotopy which measures the nonassociativity of the product m_2. So the existence of a *strictly associative* cohomology algebra $H^*(X,R)$ looks a bit strange, but although the product on $H^*(X,R)$ is associative, there appears a structure of a (generally nondegenerate) *minimal* A_∞-algebra, which can be considered as an A_∞ *deformation* of the classical cohomology $(H^*(X,R),\mu^*)$, [25].

The cohomology algebra equipped with this additional structure

$$\Big(H^*(X,R), \big\{m_i : H^*(X,R)^{\otimes i} \to H^*(X,R),\ i = 2,3,\dots\big\},\ m_1 = 0,\ m_2 = \mu^*\Big),$$

which we call cohomology A_∞-algebra, carries more information about the space than the cohomology algebra. For example just the cohomology algebra $H^*(X,R)$ does not determine the cohomology of the loop space $H^*(\Omega X,R)$, but the cohomology A_∞-algebra $(H^*(X,R),\{m_i\})$ does. Dually, the Pontriagin ring $H_*(G)$ does not determine the homology $H_*(B_G)$ of the classifying space, but the homology A_∞-algebra $(H_*(G),\{m_i\})$ does.

These A_∞-algebras have several applications in the cohomology theory of fibre bundles too, see [16].

But this invariant also is not complete. One cannot expect the existence of a more or less simple complete algebraic invariant in general, but for the rational homotopy category there are various complete homotopy invariants (algebraic models):

(i) The model of Quillen L_X, which is a differential graded Lie algebra;

(ii) The minimal model of Sullivan M_X, which is a commutative graded differential algebra;

(iii) The filtered model of Halperin and Stasheff ΛX, which is a filtered commutative graded differential algebra.

The rational cohomology algebra $H^*(X,Q)$ is not a complete invariant even for rational spaces: two spaces might have isomorphic cohomology algebras, but different rational homotopy types.

Here we also present the main result of [20]. There is the notion of C_∞-algebra which is the commutative version of Stasheff's notion of A_∞-algebra, and in [20] we have shown that in the rational case on cohomology $H^*(X,Q)$ arises a structure of C_∞-algebra $(H^*(X,Q),\{m_i\})$. The main application of this structure is the following: it completely determines the rational homotopy type, that is, 1-connected spaces X and X' have the same rational homotopy type if and only if their cohomology C_∞-algebras $(H^*(X,Q),\{m_i\})$ and $(H^*(X',Q),\{m_i'\})$ are isomorphic.

We present also several applications of this complete rational homotopy invariant to some problems of rational homotopy theory.

The C_∞-algebra structure in cohomology and the applications of this structure in rational homotopy theory were already presented in the hardly available small book [22] (see also the preprint [21]).

Applications of cohomology C_∞-algebra in rational homotopy theory are inspired by the existence of Sullivan's commutative cochains $A(X)$ in this case. The cohomology C_∞-algebra $(H^*(X,Q),\{m_i\})$ carries the same amount of information as $A(X)$ does. Actually these two objects are equivalent in the category of C_∞-algebras.

We want to remark that for simplicity in these lectures signs are ignored. Of course they can be reconstructed using the koszul sign rule.

The organization is as follows.

In Section 1 the notions of chain and cochain complexes are presented. In Section 2 the differential algebras and coalgebras are defined. In Section 3 the bar and cobar constructions are introduced. Twisting cochains and Berikashvili's functor D are presented in Section 4. In Section 5 the Stasheff's A_∞-algebras are discussed. Hochschild cochains which are used for description of A_∞-algebras are presented in Section 6. Section 7 is dedicated to our central topic, the Minimality Theorem. In the next Section 8 its applications are given. And the last Section 9 is dedicated to applications of the cohomology C_∞ algebra in rational homotopy theory.

1 Differential graded modules

1.1 Chain and cochain complexes

1.1.1 Graded modules

We work over a commutative associative ring with unit R.

A *graded module* is a collection of R-modules

$$M_* = \{\dots, M_{-1}, M_0, M_1, \dots, M_n, M_{n+1}, \dots\}.$$

A *morphism of graded modules* $M_* \to M'_*$ is a collection of homomorphisms $\{f_i: M_i \to M'_i,\ i \in Z\}$.

Sometimes we use the following notion: a *morphism of graded modules of degree* n is a collection of homomorphisms $\{f_i: M_i \to M'_{i+n},\ i \in Z\}$. So a morphism of graded modules has the degree 0.

1.1.2 Chain complexes

Definition 1.1. A differential graded (dg) module (or a chain complex) is a sequence of R modules and homomorphisms

$$\cdots \xleftarrow{d_{-1}} C_{-1} \xleftarrow{d_0} C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} \cdots \xleftarrow{d_{n-1}} C_{n-1} \xleftarrow{d_n} C_n \xleftarrow{d_{n+1}} C_{n+1} \xleftarrow{d_{n+2}} \cdots$$

such that $d_i d_{i+1} = 0$.

Elements of C_n are called *n-dimensional chains*; the homomorphisms d_i are called *boundary operators*, or *differentials*; elements of $Z_n = \operatorname{Ker} d_n \subset C_n$ are called *n-dimensional cycles* and elements of $B_n = \operatorname{Im} d_{n+1} \subset C_n$ are called *n-dimensional boundaries.*

It follows from the condition $d_i d_{i+1} = 0$ that $B_n \subset Z_n$.

Definition 1.2. The *n-th homology module* $H_n(C_*)$ of a dg module (C_*, d_*) is defined as the quotient Z_n/B_n.

A sequence $C_{n-1} \xleftarrow{d_n} C_n \xleftarrow{d_{n+1}} C_{n+1}$ is *exact*, that is, $B_n = Z_n$, iff $H_n(C_*) = 0$. Thus homology measures the deviation from exactness.

1.1.3 Cochain complexes

The notion of *cochain complex* differs from the notion of chain complex by the direction of the differential

$$\cdots \xrightarrow{d^{-1}} C^{-1} \xrightarrow{d^0} C^0 \xrightarrow{d^1} C^1 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} C^{n-1} \xrightarrow{d^n} C^n \xrightarrow{d^{n+1}} C^{n+1} \xrightarrow{d^{n+2}} \cdots .$$

Corresponding terms here are cochains, cocycles $Z^n = \operatorname{Ker} d_{n+1} \subset C^n$, coboundaries $B^n = \operatorname{Im} d^n \subset C^n$, cohomology $H^n(C^*) = Z^n/B^n$.

Changing indices $C^n = C_{-n}$, $d^n = d_{-n}$ we convert a chain complex (C_*, d_*) to a cochain complex (C^*, d^*).

1.1.4 Dual cochain complex

For a chain complex (C_*, d_*) and an R-module A the *dual cochain complex* $C^* = (\operatorname{Hom}(C_*, A), \delta^*)$ is defined as

$$C^n = (\operatorname{Hom}(C_n, A), \delta^*), \quad \delta^*(\phi) = \phi d.$$

1.1.5 Chain maps

Definition 1.3. A *chain map* of chain complexes $f : (C_*, d_*) \to (C'_*, d'_*)$ is defined as a sequence of homomorphisms $\{f_i : C_i \to C'_i\}$ such that $d'_n f_n = f_{n-1} d_n$.

This condition means the commutativity of the diagram

$$\begin{tikzcd}
\cdots & C_{n-1} \arrow[l, "d_{n-1}"'] \arrow[d, "f_{n-1}"] & C_n \arrow[l, "d_n"'] \arrow[d, "f_n"] & C_{n+1} \arrow[l, "d_{n+1}"'] \arrow[d, "f_{n+1}"] & \cdots \arrow[l, "d_{n+2}"'] \\
\cdots & C'_{n-1} \arrow[l, "d'_{n-1}"] & C'_n \arrow[l, "d'_n"] & C'_{n+1} \arrow[l, "d'_{n+1}"] & \cdots \arrow[l, "d'_{n+2}"]
\end{tikzcd} .$$

Proposition 1.1. *The composition of chain maps is a chain map.*

Chain complexes and chain maps form a category, which we denote by $DGMod$.

Proposition 1.2. *If $\{f_i\} : (C_*, d_*) \to (C'_*, d'_*)$ is a chain map, then f_n sends cycles to cycles and boundaries to boundaries, i.e.,*

$$f_n(Z_n) \subset Z'_n \ \text{ and } \ f_n(B_n) \subset B'_n.$$

Proposition 1.3. *A chain map $\{f_i\} : (C_*, d_*) \to (C'_*, d'_*)$ induces the well defined homomorphism of homology groups*

$$f_n^* : H_n(C_*) \to H_n(C'_*).$$

Homology is a functor from the category of dg modules to the category of graded modules

$$H : DGMod \to GMod.$$

1.1.6 Hom complex

For two chain complexes C, C' define the chain complex $(\operatorname{Hom}(C, C'), D)$ as

$$\operatorname{Hom}(C, C')_n = \operatorname{Hom}^n(C, C')$$

where $\operatorname{Hom}^m(C, C') = \{\phi : C_* \to C'_{*+n},\}$ is the module of homomorphisms of degree n, and the differential $D : \operatorname{Hom}^n(C, C') \to \operatorname{Hom}^{n-1}(C, C')$ is given by $D(\phi) = d'\phi + (-1)^{\deg \phi} \phi d$.

1.1.7 Tensor product

For two chain complexes A and B the tensor product $A \otimes B$ is defined as the following chain complex:

$$(A \otimes B)_n = \sum_{p+q=n} A_p \otimes B_q,$$

with differential $d_\otimes : (A \otimes B)_n \to (A \otimes B)_{n-1}$ given by

$$d_\otimes(a_p \otimes b_q) = d_p(a_p) \otimes b_q + (-1)^p a_p \otimes d'_q(b_q).$$

If $f : A \to A'$ and $g : B \to B'$ are chain maps then there is a chain map

$$f \otimes g : A \otimes B \to A' \otimes B'$$

defined as $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$.

1.1.8 Chain homotopy

Definition 1.4. Two chain maps $\{f_i\}, \{g_i\} : (C_*, d_*) \to (C'_*, d'_*)$ are called *chain homotopic*, if there exists a sequence of homomorphisms $D_n : C_n \to C'_{n+1}$,

$$\begin{tikzcd}
\cdots & C_{n-1} \arrow[l, "d_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dr, "D_{n-1}"] & C_n \arrow[l, "d_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dr, "D_n"] & C_{n+1} \arrow[l, "d_{n+1}"'] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & \cdots \arrow[l, "d_{n+2}"'] \\
\cdots & C'_{n-1} \arrow[l, "d'_{n-1}"] & C'_n \arrow[l, "d'_n"] & C'_{n+1} \arrow[l, "d'_{n+1}"] & \cdots \arrow[l, "d'_{n+2}"]
\end{tikzcd}.$$

such that $f_n - g_n = d'_{n+1} D_n + D_{n-1} d_n$. In this case we write $f \sim_D g$.

Proposition 1.4. *Chain homotopy is an equivalence relation:*

(a) $f \sim_0 f$;

(b) $f \sim_D g \Longrightarrow g \sim_{-D} f$;

(c) $f \sim_D g$, $g \sim_{D'} h \Longrightarrow f \sim_{D+D'} h$.

Proposition 1.5. *Chain homotopy is compatible with compositions:*

(a) $f \sim_D g \Longrightarrow hf \sim_{hD} hg$;

(b) $f \sim_D g \Longrightarrow fk \sim_{Dk} gk$.

Thus there is a category $hoDGMod$ whose objects are chain complexes and morphisms are chain homotopy classes

$$\mathrm{Hom}_{hoDGMod}(C, C') = [C, C'] = \mathrm{Hom}_{DGMod}(C, C')/\sim .$$

Proposition 1.6. *If two chain maps $\{f_i\}, \{g_i\} : (C_*, d_*) \to (C'_*, d'_*)$ are chain homotopic, then the induced homomorphisms of homology groups coincide:*

$$f_n^* = g_n^* : H_n(C_*) \to H_n(C'_*).$$

Thus we have the commutative diagram of functors

$$\begin{tikzcd}
DGMod \arrow[rr] \arrow[dr, "H"'] & & hoDGMod \arrow[dl, "H"] \\
& GMod &
\end{tikzcd}.$$

1.1.9 Chain equivalence

Chain complexes C and C' are called *chain equivalent* $C \sim C'$, if there exist chain maps

$$f : C \rightleftarrows C' : g$$

such that $gf \sim \mathrm{id}_C$, $fg \sim \mathrm{id}_{C'}$. This means that C and C' are isomorphic in $hoDGMod$.

A chain complex C is called *contractible* if $C \sim 0$, equivalently if $id_C \sim 0 : C \to C$.

Proposition 1.7. *Each contractible C is acyclic, i.e., $H_i(C) = 0$ for all i.*

Proposition 1.8. *If all C_i are free and C is acyclic, then C is contractible.*

1.2 Algebraic and topological examples

1.2.1 Algebraic example

Let $(A, \mu : A \otimes A \to A)$ be an associative algebra, then

$$C(A) = \left(A \xleftarrow{\mu} A \otimes A \xleftarrow{\mu\otimes id - id\otimes\mu} A \otimes A \otimes A \xleftarrow{\mu\otimes id\otimes id - id\otimes\mu\otimes id + id\otimes id\otimes\mu} \cdots \right)$$

is a chain complex: the associativity condition guarantees that $dd = 0$.

If A has a unit $e \in A$ then this complex is contractible, that is $id : C(A) \to C(A)$ and $0 : C(A) \to C(A)$ are homotopic: the suitable chain homotopy is given by $D(a_1 \otimes \cdots \otimes a_n) = (e \otimes a_1 \otimes \cdots \otimes a_n)$. This immediately implies that $C(A)$ is acyclic, that is $H_i(C(A)) = 0$ for all $i > 0$.

This example is a particular case of more general chain complex called the bar construction, see later.

1.2.2 Simplicial complexes

Simplicial complex is a formal construction, which models topological spaces.

Definition 1.5. A *simplicial complex* is a set V with a given family of finite subsets, called simplices, so that the following conditions are satisfied:

(1) all points (called *vertices*) of V are simplices;

(2) any nonempty subset of a simplex is a simplex.

A simplex consisting of $(n+1)$ points is called *n-dimensional simplex.* The 0-dimensional simplexes, i.e., the points of V are called *vertices.*

Definition 1.6. A *simplicial map* of simplicial complexes $V \to V'$ is a map of vertices $f : V \to V'$ such that the image of any simplex of V is a simplex in V'.

Proposition 1.9. *The composition of simplicial maps is a simplicial map.*

Simplicial complexes and simplicial maps form a category which we denote as SC.

Proposition 1.10. *To any simplicial set V corresponds a topological space $|V|$ (called its* realization*) and to any simplicial map $f : V \to V'$ corresponds a continuous map of realizations $|f| : |V| \to |V'|$.*

So the realization is a functor from the category of simplicial complexes to the category of topological spaces

$$|-| : SC \to Top.$$

1.2.3 Homology modules of a simplicial complex

In this section we consider *ordered* simplicial complexes: we assume that the set of vertices V is ordered by a certain order.

We assign to such an ordered simplicial complex the following chain complex $(C_*(V), d_*)$: Let $C_n(V)$ be the free R-module, generated by all ordered n-simplices $\sigma_n = (v_{k_0}, v_{k_1}, \ldots, v_{k_n})$, where $v_{k_0} < v_{k_1} < \cdots < v_{k_n}$; the differential $d_n : C_n(V) \to C_{n-1}(V)$ on a generator $\sigma_n = (v_{k_0}, v_{k_1}, \ldots, v_{k_n}) \in C_n(V)$ is given by

$$d_n(v_{k_0}, v_{k_1}, \ldots, v_{k_n}) = \sum_{i=0}^{n} (-1)^i (v_{k_0}, \ldots, \widehat{v}_{k_i}, \ldots, v_{k_n}),$$

where $(v_{k_0}, \ldots, \widehat{v}_{k_i}, \ldots, v_{k_n})$ is the $(n-1)$-simplex obtained by omitting v_{k_i}, and is extended on the whole $C_n(V)$ linearly.

Proposition 1.11. *The composition $d_{n-1}d_n$ is zero, thus $(C_*(V), d_*)$ is a chain complex.*

Definition 1.7. The *n-th homology group $H_n(V)$* of an ordered simplicial set V is defined as the n-th homology group $H_n(C_*(V))$.

1.2.4 Cohomology modules of a simplicial complex

Let A be an R-module. The cochain complex of V with coefficients in A is defined as the dual to the chain complex $C_*(V)$: $C^*(V, A) = \mathrm{Hom}(C_*(V), A)$. The n-th cohomology module of V with coefficients in A is just the n-th homology of this cochain complex.

Below we show that the cohomology $H^*(V, A)$ is more interesting than the homology $H_*(V)$, since cohomology possesses richer algebraic structure: it is a ring.

2 Differential graded algebras and coalgebras

2.1 Differential graded algebras

2.1.1 Graded algebras

A *graded algebra* is a graded module

$$A_* = \{ \ldots, A_{-1}, A_0, A_1, \ldots, A_n, A_{n+1}, \ldots \}$$

equipped with an associative multiplication $\mu : A_* \otimes A_* \to A_*$ of degree 0, i.e., $\mu(\mu \otimes id) = \mu(id \otimes \mu)$ and $\mu : A_p \otimes A_q \to A_{p+q}$. We denote $a \cdot b = \mu(a \otimes b)$.

For a graded algebra A_* the component A_0 is an associative algebra.

A *morphism of graded algebras* $f : (A, \mu) \to (A', \mu')$ is a morphism of graded modules $\{f_k : A_k \to A'_k\}$ which is multiplicative, that is, $f\mu = \mu(f \otimes f)$, i.e., $f(a \cdot b) = f(a) \cdot f(b)$.

Let $f, g : (A, \mu) \to (A', \mu')$ be two morphisms of graded algebras. An (f, g)-*derivation* of degree k is defined as a morphism of graded modules of degree k $D : A_* \to A'_{*+k}$, i.e., a collection of homomorphisms $\{D_i : A_i \to A'_{i+k},\ i \in Z\}$ which satisfies the condition

$$D(a \cdot b) = D(a) \cdot g(b) + (-1)^{k \cdot |a|} f(a) \cdot D(b).$$

An essential particular case of this notion is a *k-derivation* $D : (A, \mu) \to (A, \mu)$ which is an (id, id) *k-derivation*, i.e., it satisfies the condition $D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$.

It is easy to see that if $D, D' : A_* \to A'_{*+k}$ are two (f, g) derivation, then their sum $D + D'$ is also an (f, g) derivation.

Moreover, for graded algebra morphisms $h : B \to A$, $l : A' \to C$ and an (f, g) derivations $D : A \to A'$ the composition Dh is an (fh, gh) derivation and lD is an (lf, lg)-derivation.

2.1.2 Differential graded algebras

Definition 2.1. A differential graded algebra (*dga* in short) (A, d, μ) is a dg module (A, d) equipped additionally with a multiplication

$$\mu : A \otimes A \to A$$

so that (A, μ) is a graded algebra, and the multiplication μ is a chain map, that is, the differential d and μ are connected by the condition

$$d(a \cdot b) = da \cdot b + (-1)^{|a|} a \cdot db.$$

This condition means simultaneously that μ is a chain map, and that d is an (id, id)-*derivation* of degree -1.

A morphism of *dga*-s $f : (A, d, \mu) \to (A, d, \mu)$ is defined as a multiplicative chain map:

$$df = fd, \quad f(a \cdot b) = f(a) \cdot f(b).$$

We denote the obtained category as $DGAlg$.

For a dg algebra (A, d, μ) its homology $H_*(A)$ is a graded algebra with the following multiplication:

$$H_*(A) \otimes H_*(A) \xrightarrow{\phi} H_*(A \otimes A) \xrightarrow{H_*(\mu)} H_*(A) \ ,$$

where $\phi : H_*(A) \otimes H_*(A) \to H_*(A \otimes A)$ is the standard map

$$\phi(h_1 \otimes h_2) = cl(z_{h_1} \otimes z_{h_2}).$$

In other words the multiplication on $H(A)$ is defined as follows: For $h_1, h_2 \in H(A)$ the product $h_1 \cdot h_2$ is the homology class of the cycle $z_{h_1} \cdot z_{h_2}$.

Furthermore, a *dga* map induces a multiplicative map of homology graded algebras.

Thus homology is a functor from the category of dg algebras to the category of graded algebras.

2.1.3 Derivation homotopy

Two dg algebra maps $f, g : A \to A'$ are called homotopic if there exists a chain homotopy $D : A \to A'$, $f - g = dD + Dd$ which, in addition is a (f, g)-derivation, that is

$$D(a \cdot b) = D(a) \cdot g(b) + (-1)^{|a|} f(a) \cdot D(b).$$

Note that generally this is not an equivalence relation.

2.2 Differential graded colgebras

2.2.1 Graded coalgebras

A graded coalgebra (C, Δ) is a graded module

$$C = \{ \dots, C_{-1}, C_0, C_1, \dots, C_n, C_{n+1}, \dots \}$$

equipped with a comultiplication

$$\Delta : C \otimes C \to C$$

which is coassociative, that is $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$, i.e., the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes id"] \\ C \otimes C \arrow[r, "id \otimes \Delta"'] & C \otimes C \otimes C \end{tikzcd} .$$

A morphism of graded coalgebras $f : (C, \Delta) \to (C', \Delta')$ is a morphism of graded modules $\{f_k : C_k \to C'_k\}$ which is comultiplicative, that is $\Delta' f = (f \otimes f)\Delta$, i.e., the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta"'] & C' \arrow[d, "\Delta'"] \\ C \otimes C \arrow[r, "f\otimes f"'] & C' \otimes C' \end{tikzcd} .$$

If $f, g : C \to C'$ are two morphisms of graded coalgebras, then an (f, g)-coderivation of degree k is defined as a collection of homomorphisms $\{D_i : C_i \to C_{i+k}\}$ which satisfies the condition $\Delta' D = (f \otimes D + D \otimes g)\Delta$, i.e., the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "D"] \arrow[d, "\Delta"'] & C' \arrow[d, "\Delta'"] \\ C \otimes C \arrow[r, "f\otimes D+D\otimes g"'] & C' \otimes C' \end{tikzcd} .$$

An essential particular case of this notion is that of a *k-coderivation* $D : (C, \Delta) \to (C, \Delta)$, which is an (id, id)- *k-coderivation*, i.e., it satisfies the condition $\Delta D = (id \otimes D + D \otimes id)\Delta$.

It is easy to see that if $D, D' : C_* \to C'_{*+k}$ are two (f, g)-coderivations, then their sum $D + D'$ is also an (f, g)-coderivation.

Moreover, for graded coalgebra morphisms $h : A \to C$, $l : C' \to B$ and an (f, g)-coderivation $D : C \to C'$ the composition Dh is an (fh, gh)-coderivation and lD is an (lf, lg) coderivation.

2.2.2 Differential graded coalgebras

Definition 2.2. A differential graded coalgebra (*dgc* in short) (C, d, Δ) is a dg module (C, d) equipped additionally with a comultiplication $\Delta : C \to C \otimes C$ so that (C, Δ) is a graded coalgebra and the comultiplication Δ and the differential d are related by the condition

$$\Delta d = (d \otimes id + id \otimes d)\Delta.$$

This condition means simultaneously that Δ is a chain map, and that d is a (id, id)-coderivation of degree -1.

A morphism of *dgc*-s $f : (C, d, \Delta) \to (C', d', \Delta')$ is defined as a morphism of graded coalgebras which is a chain map.

We denote the obtained category as $DGCoalg$.

Generally for a dg coalgebra (C, d, μ) its homology $H_*(C)$ *is not* a graded coalgebra:

$$H_*(C) \otimes H_*(C) \xrightarrow{\phi} H_*(C \otimes C) \xleftarrow{H(\Delta)} H_*(C) ,$$

the map $\phi : H_*(C) \otimes H_*(C) \to H_*(C \otimes C)$ has the wrong direction, but if all $H_i(C)$ are free then ϕ is invertible and $H_*(C)$ is a graded coalgebra.

2.2.3 Coderivation homotopy

In the category of dg coalgebras there is the following notion of homotopy: two dg coalgebra maps $f, g : (C, d_C, \Delta_C) \to (C', d_{C'}, \Delta_{C'})$ are homotopic, if there exists $D : C \to C'$ of degree $+1$ such that $d_{C'}D + Dd_C = f - g$, i.e., the chain maps f and g are chain homotopic, and additionally the homotopy D is a $f - g$-coderivation, that is $\Delta_{C'}D = (f \otimes D + D \otimes g)\Delta_C$.

2.2.4 Dual cochain algebra

Let (C_*, d_C, Δ_C) be a dg coalgebra and let $(A, \mu_A : A\otimes A \to A)$ be a (nondifferential and nongraded) algebra. Then the dual cochain complex $(C^* = \mathrm{Hom}(C_*, A), \delta^*)$, $\delta^*(\phi) = \phi d_C$, becomes a dg algebra (of cochain type, i.e., the degree of the differential is $+1$) with the multiplication (cup product)

$$\mu^* : \mathrm{Hom}(C_*, A)\otimes \mathrm{Hom}(C_*, A) \to \mathrm{Hom}(C_*, A)$$

given by

$$\phi \smile \psi = \mu^*(\phi\otimes\psi) = \mu_A(\phi\otimes\psi)\Delta_C.$$

The obtained object $(C^* = \mathrm{Hom}(C_*, A), \delta^*(\phi) = \phi d_C, \smile)$ is called the cochain dg algebra of the chain dg coalgebra (C_*, d_C, Δ_C) with coefficients in A.

2.3 Applications in topology

2.3.1 Alexander–Whitney diagonal

Let V be an ordered simplicial complex (1.2.3) and let $(C_*(V), d_*)$ be its chain complex. There exists a comultiplication

$$\Delta : C_*(V) \to C_*(V)\otimes C_*(V),$$

the so called Alexander–Whitney diagonal, which turns $(C_*(V), d_*, \Delta)$ into a dg coalgebra. This diagonal is defined by

$$\Delta(v_{i_0}, \ldots, v_{i_n}) = \sum_{k=0}^{n}(v_{i_0}, \ldots, v_{i_k})\otimes(v_{i_k}, \ldots, v_{i_n}).$$

2.3.2 Cohomology algebra

Let again $(A, \mu_A : A\otimes A \to A)$ be a (nondifferential and nongraded) associative algebra. The Alexander–Whitney diagonal of $C_*(V)$ induces on the dual cochain complex

$$C^*(V, A) = \mathrm{Hom}(C^*(V), A), \delta^*)$$

the cup product

$$\smile : C^*(V)\otimes C^*(V) \to C^*(V)$$

which for $\phi\in C^p(V, A)$, $\psi \in C^q(V, A))$ looks as

$$\phi\smile\psi(v_{i_0}, \ldots, v_{i_{p+q}}) = \phi(v_{i_0}, \ldots, v_{i_p})\cdot\psi(v_{i_p}, \ldots, v_{i_{p+q}}).$$

This turns $C^*(V, A), \delta^*, \smile)$ into a dg algebra.

This structure induces on the cohomology $H^*(V, A)$ a structure of graded algebra.

The cohomology algebra $H^*(V, A)$ is a more powerful invariant than the cohomology groups: the two spaces $X = S^1\,\mathrm{times}\,S^1$ and $Y = S^1\vee S^1\vee S^2$ have the same cohomology groups

$$H^0 = R,\;\; H^1 = R\cdot a\oplus R\cdot b,\;\; H^2 = R\cdot c,$$

with generators a, b in dimension 1 and c in dimension 2, but they have different cohomology algebras, namely $a\cdot b = 0$ in $H^*(Y)$ and $a\cdot b = c$ in $H^*(X)$.

3 Bar and cobar functors

Here we describe two classical adjoint functors

$$B : DGAlg \rightleftarrows DGCoalg : \Omega\ ,$$

the bar functor $B : DGAlg \to DGCoalg$ from the category of dg algebras to the category of dg coalgebras, and the cobar functor $\Omega : DGCoalg \to DGAlg$ in the opposite direction.

We start with the definitions of free (in the category of graded algebras) and cofree (in the category of graded coalgebras) objects.

3.1 Tensor algebra and tensor coalgebra

3.1.1 Tensor algebra

Let $V = \{V_i\}$ be a graded R-module. The tensor algebra generated by V is defined as

$$T(V) = R \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots = \sum_{i=0}^{\infty} V^{\otimes i}$$

with grading $\dim(a_1 \otimes \cdots \otimes a_m) = \dim a_1 + \cdots + \dim a_m$, and with *multiplication* $\mu : T(V) \otimes T(V) \to T(V)$ given by

$$\mu\big((a_1 \otimes \cdots \otimes a_m) \otimes (a_{m+1} \otimes \cdots \otimes a_{m+n})\big) = a_1 \otimes \cdots \otimes a_{m+n}.$$

The unit element for this multiplication is $1 \in R = V^{\otimes 0}$.

By i_k we denote the obwious inclusion $i_k : V^{\otimes k} \to T(V)$.

3.1.2 Universal property of $T(V)$

The tensor algebra $T(V)$ is the free object in the category of graded algebras: for an arbitrary graded algebra (A, μ_A) and a map of graded modules $\alpha : V \to A$ there exists a unique morphism of graded algebras $f_\alpha : T(V) \to A$ such that $f_\alpha(v) = \alpha(v)$ (i.e., $f_\alpha i_1 = \alpha$).

This morphism f_α (which is called *multiplicative extension* of α) is defined as $f_\alpha(a_1 \otimes \cdots \otimes a_m) = \alpha(a_1) \cdot \cdots \cdot \alpha(a_m)$. Or, equivalently f_α is described by:

$$f_\alpha i_k = \sum_k \mu_A^k (\alpha \otimes \cdots \otimes \alpha),$$

where $\mu_A^k : A^{\otimes k} \to A$ is the k-fold iteration of the multiplication $\mu_A : A \otimes A \to A$, namely, $\mu_A^1 = id$, $\mu_A^2 = \mu_A$, $\mu_A^k = \mu_A(\mu_A^{k-1} \otimes id)$.

So, to summarize, we have the following universal property

$$\begin{tikzcd} V \arrow[r, "i_1"] \arrow[dr, "\alpha"'] & T(V) \arrow[d, "f_\alpha = \sum_k \mu_A^k(\alpha\otimes\cdots\otimes\alpha)"] \\ & A \end{tikzcd} .$$

3.1.3 Universal property for derivations

The tensor algebra has an analogous universal property also for derivations: for a graded algebra (A, μ), two homomorphisms $\alpha, \alpha' : V \to A$ of degree 0 and a homomorphism $\beta : V \to A$ of degree k there exist: morphisms of graded algebras $f_\alpha, f_{\alpha'} : T(V) \to A$ and a unique $(f_\alpha, f_{\alpha'})$ derivation of degree k

$$D_\beta : T(V) \to A$$

such that $f_\alpha(v) = \alpha(v)$, $f'_\alpha(v) = \alpha'(v)$ and $D(v) = \beta(v)$, i.e., the following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "i_1"] \arrow[dr, "\beta"'] & T(V) \arrow[d, "D"] \\ & A \end{tikzcd} .$$

The derivation D is defined as

$$D(a_1 \otimes \cdots \otimes a_n) = \sum_{k=1}^{n} \mu^n \Big(\alpha(a_1) \otimes \cdots \otimes \alpha(a_{k-1}) \otimes \beta(a_k) \otimes \alpha'(a_{k+1}) \otimes \cdots \otimes \alpha'(a_n) \Big).$$

Or, equivalently

$$D \cdot i_n = \sum_k \mu^n \big(\alpha^{\otimes(k-1)} \otimes \beta \otimes {\alpha'}^{\otimes(n-k)} \big) i_n.$$

3.1.4 Tensor coalgebra

Here is the dualization of the previous notion.

Let $V = \{V_i\}$ be a graded R-module. The *tensor coalgebra* cogenerated by V is defined (again) as

$$T^c(V) = R \oplus V \oplus V \otimes V \oplus V \otimes V \otimes V \oplus \cdots = \sum_{i=0}^{\infty} V^{\otimes i}$$

with the same grading

$$\dim(a_1 \otimes \cdots \otimes a_m) = \dim a_1 + \cdots + \dim a_m,$$

but now with *comultiplication* $\Delta : T^c(V) \to T^c(V) \otimes T^c(V)$ given by

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n),$$

here $(\ ) = 1 \in R = V^{\otimes 0}$.

By p_k we denote the obwious projection $p_k : T^c(V) \to V^{\otimes k}$.

3.1.5 Universal property of $T^c(V)$

In order to formulate the about universal property in this case we have to introduce some dimensional restrictions in this case.

Let $V = \{\ldots, 0, 0, V_1, V_2, \ldots\}$ be a *connected* graded module, that is, $V_i = 0$ for $i \leq 0$.

The tensor coalgebra of such V is the cofree object in the category of *connected* graded coalgebras: for a map of graded modules $\alpha : C \to V$ there exists a unique morphism of graded coalgebras $f_\alpha : C \to T^c(V)$ such that $p_1 f_\alpha = \alpha$, i.e., the following diagram commutes

$$\begin{tikzcd} V & T^c(V) \arrow[l, "p_1"'] \\ & C \arrow[ul, "\alpha"] \arrow[u, "f_\alpha"'] \end{tikzcd} .$$

The coalgebra map f_α (which is called comultiplicative coextension of α) is defined as

$$f_\alpha = \sum_k (\alpha \otimes \cdots \otimes \alpha)\Delta^k,$$

where $\Delta^k : C \to C^{\otimes k}$ is the k-th iteration of the comultiplication $\Delta : C \to C \otimes C$, i.e.,

$$\Delta^1 = id, \ \ \Delta^2 = \Delta, \ \ \Delta^k = (\Delta^{k-1} \otimes id)\Delta.$$

3.1.6 Universal property for coderivations

The tensor coalgebra has a similar universal property also for coderivations.

Namely, for a graded coalgebra (C, Δ), two homomorphisms of degree 0 $\alpha, \alpha' : C \to V$ and a homomorphism of degree k $\beta : C \to V$ there exist: morphisms of graded coalgebras $f_\alpha, f_{\alpha'} : C \to T(V)$ and a unique $(f_\alpha, f_{\alpha'})$-coderivation of degree k $\partial_\beta : C \to T^c(V)$ such that $p_1 \partial_\beta = \beta$, i.e., commutes the following diagram:

$$\begin{tikzcd} V & T^c(V) \arrow[l, "p_1"'] \\ & C \arrow[ul, "\beta"] \arrow[u, "\partial_\beta"'] \end{tikzcd} .$$

The coderivation ∂_β is defined as

$$\partial_\beta = \sum_{n=0}^{\infty} \sum_{k=1}^{n} (\alpha^{\otimes(k-1)} \otimes \beta \otimes {\alpha'}^{(n-k)} \Delta^k.$$

3.2 Shuffle comultiplication and shuffle comultiplication – bialgebra structures on $T(V)$ and $T'(V)$

In fact, $T(V)$ and $T^c(V)$ coincide as graded modules, but the multiplication of $T(V)$ and the comultiplication of $T^c(V)$ are not compatible with each other, so they do not define a graded bialgebra structure on the graded module $T(V) = T^c(V)$.

Nevertheless there exists the *shuffle* comultiplication

$$\nabla_{sh} : T(V) \to T(V) \otimes T(V)$$

introduced by Eilenberg and MacLane [7], which turns $(T(V), \Delta_{sh}, \mu)$ into a graded bialgebra.

And dually, there exists the *shuffle* multiplication $\mu_{sh} : T^c(V) \otimes T^c(V) \to T^c(V)$ which turns $(T^c(V), \Delta, \mu_{sh})$ into a graded bialgebra.

3.2.1 Shuffle comultiplication – bialgebra structure on tensor algebra

There exists on this free graded algebra $(T(V), \mu)$ a comultiplication $\nabla_{sh} : T(V) \to T(V) \otimes T(V)$ which is a morphism of graded algebras, and, consequently

turns $(T(V), \mu, \nabla_{sh})$ into a graded bialgebra.

Namely, the *shuffle comultiplication* $\nabla_{sh} : T(V) \to T(V) \otimes T(V)$ is a graded coalgebra map induced by the universal property of $T(V)$ (3.1.2) by $\alpha : T(V) \to T(V) \otimes T(V)$ given by $\alpha(v) = v \otimes 1 + 1 \otimes v$, $\alpha(1) = 1 \otimes 1$.

This comultiplication is associative and in fact is given by

$$\begin{aligned}\nabla(v_1 \otimes \cdots \otimes v_n) = 1 \otimes (v_1 \otimes \cdots \otimes v_n) + (v_1 \otimes \cdots \otimes v_n) \otimes 1 \\ + \sum_p \sum_{\sigma \in sh(p,n-p)} (v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)}) \otimes (v_{\sigma(p+1)} \otimes \cdots \otimes v_{\sigma(n)})\end{aligned}$$

where $sh(p, n-p)$ consists of all $(p, n-p)$-shuffles, that is all permutations of $1, \ldots, n$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(n)$.

3.2.2 Shuffle multiplication – bialgebra structure on tensor coalgebra

The *shuffle* multiplication $\mu_{sh} : T^c(V) \otimes T^c(V) \to T^c(V)$, introduced by Eilenberg and MacLane [7], turns $(T^c(V), \Delta, \mu_{sh})$ into a graded bialgebra.

This multiplication is defined as the graded coalgebra map induced by the universal property of $T^c(V)$ (3.1.5) by $\alpha : T^c(V) \otimes T^c(V) \to V$ given by $\alpha(v \otimes 1) = \alpha(1 \otimes v) = v$ and $\alpha = 0$ otherwise. This multiplication is associative and in fact is given by

$$\mu_{sh}\big((v_1 \otimes \cdots \otimes v_n) \otimes (v_{n+1} \otimes \cdots \otimes v_{n+m})\big) = \sum \pm v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n+m)},$$

where summation is taken over all (m, n)-shuffles, that is, over all permutations of the set $(1, 2, \ldots, n+m)$ which satisfy the condition: $i < j$ if $1 \le \sigma(i) < \sigma(j) \le n$ or $n+1 \le \sigma(i) < \sigma(j) \le n+m$.

Particularly,

$$\begin{gathered}\mu_{sh}((v_1) \otimes (v_2)) = v_1 \otimes v_2 \pm v_2 \otimes v_1, \\ \mu_{sh}((v_1) \otimes (v_2 \otimes v_3)) = v_1 \otimes v_2 \otimes v_3 \pm v_2 \otimes v_1 \otimes v_3 \pm v_2 \otimes v_3 \otimes v_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{gathered}$$

3.3 Bar and cobar constructions

3.3.1 Cobar construction

Let (C, d, Δ) be a dg coalgebra with $C_i = 0$, $i \le 1$ and let $s^{-1}C$ be the desuspension of C, that is, $(s^{-1}C)_k = C_{k+1}$.

The cobar construction ΩC is defined as the tensor algebra $T(s^{-1}C)$. We use the following notation for elements of this tensor coalgebra:

$$s^{-1}a_1 \otimes \cdots \otimes s^{-1}a_n = [a_1, \dots, a_n].$$

So the dimension of $[a_1, \dots, a_n]$ is $\sum_i \dim a_i - n$.

The differential $d_\Omega : \Omega C \to \Omega C$ is defined as

$$d_\Omega[a_1, \dots, a_n] = \sum_i \pm\big[a_1, \dots, a_{i-1}, da_i, a_{i+1}, \dots, a_n\big] + \sum_i \pm\big[a_1, \dots, a_{i-1}, \Delta(a_i), a_{i+1}, \dots, a_n\big].$$

In fact $d_\Omega = \partial_\beta$ where ∂_β is the derivation defined by the above universal property (3.1.3) for

$$\beta[a] = da + \Delta a.$$

Besides, the properties of d and Δ from the definition of a dg coalgebra (2.2.2) guarantee that the restriction $d_\Omega d_\Omega|V = d_\Omega d_\Omega i_1$ is 0 and this, by the universal property (3.1.3), implies $d_\Omega d_\Omega = 0$. Thus $\Omega C \in DGAlg$.

3.3.2 Bar construction

Let $(A, d, \mu = \cdot)$ be a dg algebra with $A_i = 0$ for $i \leq 1$ and let sA be the suspension of A, that is $(sA)_k = A_{k-1}$.

As a graded coalgebra the bar construction BA is defined as the tensor coalgebra: $T^c(sA)$. We use the following notation for elements of this tensor coalgebra

$$sa_1 \otimes \cdots \otimes sa_n = [a_1, \dots, a_n].$$

So the dimension of $[a_1, \dots, a_n]$ is $\sum_i \dim a_i + n$.

The differential $d_B : BA \to BA$ is defined as

$$d_B[a_1, \dots, a_n] = \sum_i \pm\big[a_1, \dots, a_{i-1}, da_i, a_{i+1}, \dots, a_n\big] + \sum_i \pm\big[a_1, \dots, a_i \cdot a_{i+1}, \dots, a_n\big].$$

In fact, $d_B = D_\beta$ where D_β is the coderivation defined by the above universal property (3.1.6) for

$$\beta[a_1, \dots, a_n] = \begin{cases} [da_1] & \text{for } n = 1; \\ [a_1 \cdot a_2] & \text{for } n = 2 \\ 0 & \text{for } n > 2. \end{cases}$$

Besides, the properties of d and μ from the definition of a dg algebra (2.1.2) guarantee that the projection $p_1 d_B d_B$ is 0 and this, by the universal property (3.1.6), implies $d_B d_B = 0$. Thus $BA \in DGCoalg$.

3.3.3 Bar construction of a commutative dg algebra

Assume now that $(A, d, \mu = \cdot)$ is a commutative dg algebra. How the commutativity reflects on the bar construction BA?

By definition, the differential $d_B : BA \to BA$ is a coderivatian with respect to the standard graded coalgebra structure of the tensor coalgebra $BA = T^c(sA)$. As it was mentioned above in (3.2.2), BA carries also the shuffle product $\mu_{sh} : T^c(sV) \otimes T^c(sA) \to T^c(sA)$ which turns it into a graded bialgebra.

Proposition 3.1. *If a dg algebra $(A, d, \mu = \cdot)$ is commutative, the differential $d_\beta : BA \to BA$ is not only a coderivation with respect to comultiplication Δ but also a derivation with respect to the shuffle product μ_{sh}. So in this case the bar construction (BA, Δ, μ_{sh}) is a dg bialgebra.*

Proof. The map $\Phi : BA \otimes BA \to BA$ defined as

$$\Phi = d_\beta \mu_{sh} - \mu_{sh}(d_\beta \otimes id + id \otimes d_\beta)$$

is a coderivation, see the arguments in (2.2.1). Thus, according to universal the property (3.1.6) of $T^c(s^{-1}A)$, the map Φ is trivial if and only if $p_1\Phi = 0$. This is equivalent to the commutativity of A. □

3.3.4 Adjunction

Let (C, d, Δ) be a dg coalgebra and (A, d, μ) a dg algebra. A twisting cochain [5] is a homomorphism $\tau : C \to A$ of degree $+1$ satisfying Browns' condition

$$d\tau + \tau d = \tau \smile \tau, \tag{3.1}$$

where $\tau \smile \tau' = \mu_A(\tau \otimes \tau')\Delta$. We denote by $T(C, A)$ the set of all twisting cochains $\tau : C \to A$.

There are universal twisting cochains $C \to \Omega C$ and $BA \to A$, namely, the obvious inclusion and projection, respectively. Here are essential consequences of the condition (4.1):
(i) The multiplicative extension $f_\tau : \Omega C \to A$ is a map of dg algebras, so there is a bijection $T(C, A) \longleftrightarrow \mathrm{Hom}_{DG-Alg}(\Omega C, A)$;
(ii) The comultiplicative coextension $f_\tau : C \to BA$ is a map of dg coalgebras, so there is a bijection $T(C, A) \longleftrightarrow \mathrm{Hom}_{DG-Coalg}(C, BA)$.

Thus we have two bijections

$$\mathrm{Hom}_{DG-Alg}(\Omega C, A) \longleftrightarrow T(C, A) \longleftrightarrow \text{``}\mathrm{Hom}_{DG-Coalg}(C, BA).$$

Besides, there are two weak equivalences (homology isomorphisms)

$$\alpha_A : \Omega B(A) \to A, \quad \beta_C : C \to B\Omega(C).$$

4 Twisting cochains and functor D

4.1 Brown's twisting cochains

4.1.1 Definition of twisting cochain

Let $(K, d_K : K_* \to K_{*-1}, \nabla_K : K \to K \otimes K)$ be a dg coalgebra and let $(A, d_A : A_* \to A_{*-1}, \mu_A : A \otimes A \to A)$ be a dg algebra. Then $C^*(K, A) = \mathrm{Hom}(K, A)$ with differential $\delta\alpha = \alpha d_K + d_A\alpha$ and multiplication $\alpha \smile \beta = \mu_A(\alpha \otimes \beta)\nabla_K$ is a dg algebra.

Definition 4.1. A Brown twisting cochain is a homomorphism

$$\phi : K_* \to A_{*-1},$$

i.e., $deg\phi = -1$, satisfying

$$d\phi + \phi d = \phi \smile \phi. \tag{4.1}$$

Brown's condition sometimes is called also *Maurer-Cartan equation*. It can be rewritten as $d_A\phi + \phi d_K = \mu_A(\phi \otimes \phi)\nabla_K$.

The set of all twisting cochains $\phi : K \to A$ we denote as $Tw(K, A)$.

4.1.2 Twisted tensor product

Let $(M, d_M, \nu : A \otimes M \to M)$ be a dg A-module. Then any twisting cochain $\phi : K \to A$ determines a homomorphism $d_\phi : K \otimes M \to K \otimes M$ by

$$d_\phi(k \otimes m) = d_K k \otimes m + k \otimes d_M m + (k \otimes m) \cap \phi$$

where

$$(k \otimes m) \cap \phi = (id_K \otimes \nu)(id_K \otimes \phi \otimes id_M)(\nabla_K \otimes id_M)(k \otimes m).$$

Brown's condition $d\phi = \phi \smile \phi$ implies that $d_\phi d_\phi = 0$.

Definition 4.2. The twisted tensor product $(K \otimes_\phi M, d_\phi)$ is defined as the chain complex $(K \otimes M, d_\phi)$.

4.1.3 Application: model of fibration

Let (E, p, B, F, G) be a fibre bundle with base B, fibre F, structure group G. So $K = C_*(B)$ is a dg coalgebra, $A = C_*(G)$ is a dg algebra, $M = C_*(F)$ is a dg module over A. Then, by Brown's theorem there exists a twisting cochain $\phi : K = C_*(B) \to A = C_*(G)$ such that the twisted tensor product

$$(K \otimes_\phi M, d_\phi) = \left(C_*(B) \otimes_{\phi : C_*(B) \to C_{*-1}(G)} C_*(F), d_\phi\right)$$

gives the homology of fibre space $H_*(E)$. It is clear that ϕ determines all differentials of the Serre spectral sequence also.

Brown's twisting cochain ϕ is not determined uniquely, so it would be useful to have a possibility to choose one convenient for computations.

4.1.4 Twisting cochains and the Bar and cobar constructions

By the universal property of the tensor coalgebra from (B1.3) any homomorphism $\phi : K \to A$ of degree -1 induces a graded coalgebra map $f_\phi : K \to T^c(sA)$ by

$$f_\phi = \sum_i (\phi \otimes \cdots \otimes \phi)\nabla_K^i.$$

If, in addition, $\phi : K \to A$ is a twisting cochain, that is, it satisfies Brown's condition $\delta\phi = \phi \smile \phi$, then $f_\phi : K \to B(A)$ is a chain map, i.e., is a map of dg coalgebras: since the tensor coalgebra is cofree one has the equality $d_B f_\phi = f_\phi d_K$ if and only if equal the projections $p d_B f_\phi$ and $= p f_\phi d_K$ are equal, and this is exactly Brown's condition $\delta\phi = \phi \smile \phi$.

Conversely, any dg coalgebra map $f : K \to BA$ is f_ϕ for $\phi = pf : K \to BA \to A$. In fact we have a bijection $Mor_{DGCoalg}(K, BA) \longleftrightarrow Tw(K, A)$.

Dually, by the universal property of the tensor algebra (B1.2), any homomorphism $\phi : K \to A$ of degree -1 induces a graded algebra map $g_\phi : T(s^{-1}K) \to A$ by

$$g_\phi = \sum_k \mu^k(\alpha \otimes \cdots \otimes \alpha).$$

If, in addition, $\phi : K \to A$ is a twisting cochain, that is, it satisfies Brown's condition $\delta\phi = \phi \smile \phi$, then $g_\phi : \Omega K \to A$ is a chain map, i.e., it is a map of dg algebras: since of freeness of tensor algebra one has the equality $d_A g_\phi = g_\phi d_\Omega$ if and only if equal the restrictions $d_A g_\phi i = g_\phi d_\Omega i$ and this is exactly the Brown's condition $\delta\phi = \phi \smile \phi$.

Conversely, any dg algebra map $g : \Omega K \to A$ is g_ϕ for $\phi = gi : K \to \Omega K \to A$. In fact we have a bijection $Mor_{DGAlg}(\Omega K, A) \longleftrightarrow Tw(K, A)$.

So we have bijections

$$\begin{tikzcd} & Tw(K,A) \arrow[dl, "\approx"'] \arrow[dr, "\approx"] & \\ Mor_{DGAlg}(\Omega K, A) & & Mor_{DGCoalg}(K, BA) \end{tikzcd} .$$

4.1.5 (Co)universal twisting cochains

The standard inclusion $i : K \to \Omega K$ satisfies Brown's condition. So it is a (so called universal) twisting cochain. Thus it induces two chain maps

$$f_i : \Omega K \to \Omega K \ \text{ and } \ g_i : K \to B\Omega K.$$

The first one is the identity map and the second one a quasi isomorphism (homology isomorphism).

Dually, the standard projection $p : BA \to A$ satisfies the Brown's condition, so it is a (so called couniversal) twisting cochain. Thus it induces two chain maps

$$f_p : \Omega BA \to A \ \text{ and } \ g_p : BA \to BA.$$

The first one is a quasi isomorphism (homology isomorphism) and the second one is the identity map.

4.2 Berikashvili's functor D

4.2.1 Equivalence of twisting cochains

Two twisting cochains $\phi, \psi : K \to A$ are equivalent (Berikashvili [3]) if there exists $c : K \to A$, $\deg c = 0$, $c = \Sigma_i c_i$, $c_i : C_i \to A_i$ with $c_0 = c|C_0 = 0$, such that

$$\psi = \phi + \delta c + \psi \smile c + c \smile \phi, \tag{4.2}$$

notation $\phi \sim_c \psi$.

This is an equivalence relation:

$$\phi \sim_{c=0} \phi; \quad \phi \sim_c \phi', \;\; \phi' \sim_{c'} \phi'' \Longrightarrow \phi \sim_{c+c'+c' \smile c} \phi''$$

and $\phi \sim_c \phi' \Longrightarrow \phi' \sim_{c'} \phi$ where c' can be solved from $c + c' + c' \smile c = 0$ inductively.

This notion of equivalence allows to *perturb* twisting cochains. Let

$$\phi = \phi_2 + \phi_3 + \cdots + \phi_n + \cdots : K \to A, \;\; \phi_n : K_n \to A_{n-1}$$

be a twisting cochain, and let's take an arbitrary cochain $c = c_n : K_n \to A_n$. Then there exists a twisting cochain $F_{c_n}\phi = \psi : K \to A$ such that $\phi \sim_{c_n} \psi$. Actually the components of the perturbed twisting cochain

$$F_{c_n}\phi = \psi = \psi_2 + \psi_3 + \cdots + \psi_n + \cdots$$

can be solved from (4.2) inductively, and the solution particularly gives that the perturbation $F_{c_n}\phi$ does not change the first components, i.e. $\psi_i = \phi_i$ for $i < n$ and $\psi_n = \phi_n + d_A c_n$.

The main benefit of this notion is that the equivalent twisting cochains define isomorphic twisted tensor products:

Theorem 4.1. *If $\phi \sim_c \psi$ then*

$$(K \otimes_\phi M, d_\phi) \xrightarrow{F_c} (K \otimes_\psi M, d_\psi)$$

given by $F_c(k \otimes m) = (k \otimes m) + (k \otimes m) \cap c$ is an isomorphism of dg modules.

The inverse $(K \otimes_\psi M, d_\psi) \xrightarrow{F_{c'}} (K \otimes_\phi M, d_\phi)$ is defined by $c' : K \to A$ which, as above, can be solved inductively from $c + c' + c' \smile c = 0$.

Let

$$Tw(K, A) = \{\phi : 3K \to A, \; \delta\phi = \phi \circ \phi\}$$

be the set of all twisting cochains.

Definition 4.3. Berikasvili's functor $D(K, A)$ is defined as the factorset $D(K, A) = \frac{Tw(K,A)}{\sim}$.

Berikashvili's relation of equivalence of twisting cochains allows to perturb a given twisting cochain so as to get the simplest one to simplify calculations.

4.2.2 Equivalence of twisting cochains and homotopy of induced maps

How does the equivalence of twistimg cochains affect induced maps $K \to BA$ and $\Omega K \to A$?

Theorem 4.2. *If $\phi \sim_c \psi$ then f_ϕ and f_ψ are homotopic as dg coalgebra maps: the coderivation chain homotopy $D(c) : K \to BA$ is given by*

$$D(c) = \sum_{i,j} \Big(\psi \otimes \cdots (j \;\text{ times}) \cdots \otimes \psi \otimes c \otimes \phi \otimes \cdots \otimes \phi\Big) \nabla_K^i. \tag{4.3}$$

Proof. The homotopy $D(c)$ satisfies $f_\phi - f_\psi = d_B D(c) + D(c) d_K$ because the projection of this condition on A gives $pf_\phi - pf_\psi = pd_B D(c) + pD(c)d_K$ and this is exactly the condition (4.2). Besides, $D(c)$ is a $f_\phi - f_\psi$-coderivation, that is,

$$\nabla_B D(c) = (f_\psi \otimes D(c) + D(c) \otimes f_\phi)\nabla_K.$$

This also follows from the universal property of the tensor coalgebra and the extension rule (4.3).

The converse is also true: if f_ϕ and f_ψ are homotopic by a coderivation homotopy $D : K \to BA$ then $\phi \sim_c \psi$ by $c = p_1 D$.

The dual statement is also true: g_ϕ, $g_\psi : \omega K \to A$ are homotopic in the category of dg algebras if and only if $\phi \sim \psi$.

So we have bijections

$$\begin{array}{ccc} & D(K,A) & \\ \swarrow^{\approx} & & \searrow^{\approx} \\ [\Omega K, A]_{DGAlg} & & [(K, BA)]_{DGCoalg} \end{array}$$

this means that B and Ω are adjoint functors. □

4.2.3 Lifting of twisting cochains

Any dg algebra map $f : A \to A'$ induces the map $Tw(K,A) \to Tw(K,A')$: if $\phi : K \to M$ is a twisting cochain so is the composition $f\phi : K \to A \to A'$. Moreover, if $\phi \sim_c \phi'$ then $f\phi \sim_{fc} f\phi'$. Thus we have a map $D(f) : D(K,A) \to D(K,A')$.

Theorem 4.3 (Berikashvili [3]). *Let (K, d_K, ∇_K) be a dg colagebra with free K_i and let (A, d_A, μ_A) be a connected dg algebra. If $f : A \to A'$ is a weak equivalence of connected dg algebras (i.e., a homology isomorphism), then*

$$D(f) : D(K,A) \to D(K,A')$$

is a bijection.

Particularly this theorem means that $[K, BA] \to [K, BA']$ is a bijection.

Below we'll need the surjectivity part of this theorem whose proof we sketch here.

Theorem 4.4. *Let (K, d_K, ∇_K) be a dg colagebra with free K_is and let $f : (A, d_A, \mu_A) \to (A', d_{A'}, \mu_{A'})$ be a weak equivalence of connected dg algebras. Then for an arbitrary twisting cochain*

$$\phi = \phi_2 + \phi_3 + \cdots + \phi_n + \cdots : K \to A'$$

there exists a twisting cochain

$$\psi = \psi_2 + \psi_3 + \cdots + \psi_n + \cdots : K \to A$$

such that $\phi \sim f\psi$.

Proof. Start with a twisting cohain $\phi = \phi_2 + \phi_3 + \cdots + \phi_n + \cdots : K \to A'$. Brown's defining condition (4.1) gives $d_A \phi_2 = 0$, and since $f : A \to A'$ is a homology isomorphism there exist $\psi_2 : K_2 \to A_1$ and $c_2' : K_2 \to A_2'$ such that $d_A \psi_2 = 0$ and $f\psi_2 = \phi_2 + d_{A'} c_2'$ (we assume all K_n are free modules). Perturbing ϕ by this c_2' we obtain a new twisting cochain $F_{c_2'}\phi$ for which $(F_{c_2'}\phi)_2 = \phi_2 + d_{A'} c_2' = f\psi_2$. So we can assume that $\phi_2 = f\psi_2$.

Assume now that we already have $\psi_2, \psi_3, \ldots, \psi_{n-1}$ which satisfy (4.1) in appropriate dimensions and $f\psi_k = \phi_k$, $k = 2, 3, \ldots, n-1$. We need the next component $\psi_n : K_n \to A_{n-1}$ such that

$$d_A \psi_n = \psi_{n-1} d_K + \sum_{i=2}^{n-2} \psi_i \smile \psi_{n-i}$$

and $c'_n : K_n \to A'_n$ such that $f\psi_n = \phi_n + d_{A'}c'_n$. Then perturbing ϕ by c'_n we obtain new ϕ_n for which $f\psi_n = \phi_n$, and this will complete the proof.

Let us write

$$U_n = \psi_{n-1}d_K + \sum_{i=2}^{n-2} \psi_i \smile \psi_{n-i} : K_n \to A_{n-2},$$

$$U'_n = \phi_{n-1}d_K + \sum_{i=2}^{n-2} \phi_i \smile \phi_{n-i} : K_n \to A'_{n-2}.$$

So we have $U'_n = d_{A'}\phi_n$ and we want $\psi_n : K_n \to A_{n-1}$, $c'_n : K_n \to A'_n$ such that $U_n = d_A\psi_n$ and $f\psi_n = \phi_n + d_{A'}c'_n$.

First, it is not hard to check that $d_AU_n = 0$, that is, U_n maps K_n to cycles $Z(A_{n-2}) \subset A_{n-2}$.

Then

$$\begin{aligned} fU_n &= f(\psi_{n-1}d_K + \sum_{i=2}^{n-2} \psi_i \smile \psi_{n-i}) \\ &= f\psi_{n-1}d_K + \sum_{i=2}^{n-2} f\psi_i \smile f\psi_{n-i} = \phi_{n-1}d_K + \sum_{i=2}^{n-2} \phi_i \smile \phi_{n-i} = U'_n = d_A\phi_n, \end{aligned}$$

thus, fU_n maps K_n to boundaries $B(A_{n-2}) \subset A_{n-2}$. Since $f : A \to A'$ is a homology isomorphism, there is $\overline{\psi}_n : K_n \to A_{n-1}$ such that $d_A\overline{\psi}_n = U_n$.

Now we must take care of the condition $f\psi_n = \phi_n$. It is clear that $d_Af\overline{\psi}_n = d_A\phi_n$. Let us denote by $z'_n = f\overline{\psi}_n - \phi_n$; this is the homomorphism which maps K_n to cycles $Z(A'_{n-1})$. Again, since $f : A \to A'$ is a homology isomorphism there exist $z_n : K_n \to Z(A_{n-1})$ and $c'_n : K_n \to A'_n$ such that $fz_n = z'_n - d_{A'}c'_n$. Let us define $\psi_n = \overline{\psi}_n - z_n$. Then $d_A\psi_n = d_A\overline{\psi}_n$ and

$$f\psi_n = f\overline{\psi}_n - fz_n = (\phi_n + z'_n) - (z'_n - d_{A'}c'_n) = \phi_n + d_{A'}c'_n.$$

Perturbing ϕ by this c'_n we obtain $F_{c'_n}\phi$ with $(F_{c'_n}\phi)_n = \phi_n + d'_Ac'_n = f\psi_n$. □

5 Stasheff's A_∞-algebras

5.1 Category of A_∞-algebras

The notion of A_∞-algebra was introduces by J. Stasheff [31]. This notion generalizes the notion of differential graded algebra and in fact it is so called *strong homotopy associative* algebra where the strict associativity is replaced with *associativity up to higher coherent homotopies.*

5.1.1 Notion of A_∞-algebra

Definition 5.1. An A_∞-algebra is a graded module $M = \{M^k\}_{k\in Z}$ equipped with a sequence of operations

$$\{m_i : M \otimes \cdots (i \text{ times}) \cdots \otimes M \to M,\ i = 1, 2, 3, \dots\}$$

satisfying the conditions $m_i((\otimes^i M)^q) \subset M^{q-i+2}$, that is $\deg m_i = 2 - i$, and

$$\sum_{k=0}^{i-1}\sum_{j=1}^{i-k} \pm m_{i-j+1}\big(a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes \cdots \otimes a_i\big) = 0. \tag{5.1}$$

For $i = 1$ this condition reads

$$m_1m_1 = 0.$$

For $i = 2$ this condition reads

$$m_1m_2(a_1 \otimes a_2) \pm m_2(m_1(a_1) \otimes a_2) \pm m_2(a_1 \otimes m_1(a_2)) = 0.$$

For $i = 3$ this condition reads

$$m_1m_3(a_1 \otimes a_2 \otimes a_3) \pm m_3(m_1(a_1) \otimes a_2 \otimes a_3) \pm m_3(a_1 \otimes m_1(a_2) \otimes a_3) \\ \pm m_3(a_1 \otimes a_2 \otimes m_1(a_3)) \pm m_2(m_2(a_1 \otimes a_2) \otimes a_3) \pm m_2(a_1 \otimes m_2(a_2 \otimes a_3)) = 0.$$

These three conditions mean that for an A_∞-algebra $(M, \{m_i\})$ the first two operations form a *nonassociative dga* (M, m_1, m_2) with differential m_1 and multiplication m_2 which is associative just up to homotopy and the suitable homotopy is the operation m_3.

Special case: An A_∞-algebra $(M, \{m_1, m_2, m_3 = 0, m_4 = 0, \dots\})$ is a strictly associative dg algebra.

5.1.2 Morphism of A_∞-algebras

This is the notion from [16].

Definition 5.2. A morphism of A_∞-algebras

$$\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$$

is a sequence $\{f_i : \otimes^i M \to M',\ i = 1, 2, \dots,\ \deg f_1 = 1 - i\}$ such that

$$\sum_{k=0}^{i-1}\sum_{j=1}^{i-k} \pm f_{i-j+1}\big(a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes \cdots \otimes a_i\big) \\ = \sum_{t=1}^{i} \sum_{k_1+\cdots+k_t=i} \pm m'_t\big(f_{k_1}(a_1 \otimes \cdots \otimes a_{k_1}) \otimes \cdots \otimes f_{k_t}(a_{i-k_t+1} \otimes \cdots \otimes a_i)\big). \quad (5.2)$$

In particular for $n = 1$ this condition gives $f_1m_1(a) = m'_1f_1(a)$, i.e., $f_1 : (M, m_1) \to (M', m'_1)$ is a chain map; for $n = 2$ it gives

$$f_1m_2(a_1 \otimes a_2) + m'_2(f_1(a_1) \otimes f_1(a_2)) = m'_1f_2(a_1 \otimes a_2) + f_2(m_1a_1 \otimes a_2) + f_2(a_1 \otimes m_1a_2),$$

thus $f_1 : (M, m_1, m_2) \to (M', m'_1, m'_2)$ is multiplicative just up to the chain homotopy f_2.

The composition of A_∞ morphisms

$$\{h_i\} : (M, \{m_i\}) \xrightarrow{\{f_i\}} (M', \{m'_i\}) \xrightarrow{\{g_i\}} (M'', \{m''_i\})$$

is defined as

$$h_n(a_1 \otimes \cdots \otimes a_n) = \sum_{t=1}^{n} \sum_{k_1+\cdots+k_t=n} g_n\big(f_{k_1}(a_1 \otimes \cdots \otimes a_{k_1}) \otimes \cdots \otimes f_{k_t}(a_{n-k_t+1} \otimes \cdots \otimes a_n)\big). \quad (5.3)$$

The bar construction argument (see (5.1.3) below) allows to show that so defined composition satisfies the condition (5.2).

Special case: A morphism of A_∞-algebras

$$\{f_1, f_2 = 0, f_3 = 0, \dots\} : \big(M, \{m_1, m_2, m_3 = 0, m_4 = 0, \dots\}\big) \to \big(M, \{m_1, m_2, m_3 = 0, m_4 = 0, \dots\}\big)$$

e is ordinary map of *DG*-algebras. In fact, the category of dg algebras is a subcategory of the category of A_∞-algebras.

5.1.3 Bar construction of an A_∞-algebra

Let $(M, \{m_i\})$ be an A_∞-algebra. The structure maps m_i define the map $\beta : T^c(s^{-1}M) \to s^{-1}M$ by $\beta[a_1, \dots, a_n] = [s^{-1}m_n(a_1 \otimes \cdots \otimes a_n)]$. Extending this β as a coderivation (see the universal property (3.1.6)) we obtain $d_\beta : T^c(s^{-1}M) \to T^c(s^{-1}M)$ which in fact looks as

$$d_\beta[a_1, \dots, a_n] = \sum_k \pm\big[a_1, \dots, a_k, m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}), a_{k+j+1}, \dots a_n\big].$$

The defining condition (5.1) of an A_∞-algebra guarantees that $d_\beta d_\beta = 0$: the composition of coderivations $d_\beta d_\beta : T^c(s^{-1}M) \to T^c(s^{-1}M)$ is a coderivation and the defining condition (5.1) is nothing else than that $p_1 d_\beta d_\beta = 0$ and this is equivalent to $d_\beta d_\beta = 0$ since the tensor coalgebra is cofree.

The obtained dg coalgebra $(T^c(s^{-1}M), d_\beta, \Delta)$ is called *bar construction* of the A_∞-algebra $(M, \{m_i\})$ and is denoted by $B(M, \{m_i\})$.

For an A_∞-algebra of type $(M, \{m_1, m_2, 0, 0, \dots\})$, i.e., for a strictly associative dg algebra, this bar construction coincides with the ordinary bar construction of this dg algebra.

5.1.4 Bar interpretation of a morphism of A_∞-algebras

A morphism of A_∞-algebras $\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ defines a dg coalgebra map of bar constructions

$$F = B(\{f_i\}) : B(M, \{m_i\}) \to B(M', \{m'_i\})$$

as follows: The collection $\{f_i\}$ defines the map $\alpha : T^c(s^{-1}M) \to s^{-1}M$ by

$$\alpha[a_1, \dots, a_n] = \big[s^{-1}f_n(a_1 \otimes \cdots \otimes a_n)\big].$$

Extending this α as a coalgebra map (3.1.2) we obtain $F : T^c(s^{-1}M) \to T^c(s^{-1}M)$ which in fact looks like

$$F[a_1, \dots, a_n] = \sum \pm\big[f_{k_1}(a_1 \otimes \cdots \otimes a_{k_1}), \dots, f_{k_t}(a_{n-k_t+1} \otimes \cdots \otimes a_n)\big].$$

The defining condition (5.2) of an A_∞-morphism is nothing else than $p_1 d_{\beta'} F = p_1 F d_\beta$, and this is equivalent to $d_{\beta'} F = F d_\beta$, so F is a chain map.

Now we are able to show that the composition of A_∞-morphisms is well defined: to the composition of morphisms (5.3) corresponds the composition of dg coalgebra maps

$$B((M, \{m_i\})) \xrightarrow{B(\{f_i\})} B((M', \{m'_i\})) \xrightarrow{B(\{g_i\})} B((M'', \{m''_i\}))$$

which is a dg coalgebra map. Thus for the projection $p_1 B(\{g_i\}) B(\{f_i\})$, i.e., for the collection $\{h_i\}$, the condition (5.2) is satisfied.

5.1.5 Homotopy in the category of A_∞-algebras

Two morphisms of A_∞-algebras $\{f_i\}, \{g_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ we call *homotopic* if there exists a collection of homomorphisms $\{h_i : (\otimes^i M) \to M',\ i = 1, 2, \dots,\ \deg h_i = -i\}$, which satisfy the following condition

$$\begin{aligned} &f_n(a_1 \otimes \cdots \otimes a_n) - g_n(a_1 \otimes \cdots \otimes a_n) \\ &\quad= \sum_{i+j=n+1} \sum_{k=0}^{n-j} h_i\big(a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes \cdots \otimes a_n\big) \\ &\quad+ \sum_{k_1+\cdots+k_t=n} m'_t\Big(f_{k_1}(a_1 \otimes \cdots \otimes a_{k_1}) \otimes \cdots \otimes f_{k_{i-1}}(a_{k_1+\cdots k_{i-2}+1} \otimes \cdots \otimes a_{k_1+\cdots k_{i-1}}) \\ &\qquad \otimes h_{k_i}(a_{k_1+\cdots+k_{i-1}+1} \otimes \cdots \otimes a_{k_1+\cdots+k_i}) \\ &\qquad \otimes g_{k_{i+1}}(a_{k_1+\cdots k_i+1} \otimes \cdots \otimes a_{k_1+\cdots k_{i+1}}) \otimes g_{k_t}(a_{k_1+\cdots k_{t-1}+1} \otimes \cdots \otimes a_{k_1+\cdots+k_t})\Big). \end{aligned} \tag{5.4}$$

In particular for $n = 1$ this condition means

$$f_1(a) - g_1(a) = m_1' h_1(a) + h_1(m_1 a),$$

that is, the chain maps $f_1, g_1 : (M, m_1) \to (M', m_1')$ ar chain homotopic.

For $n = 2$ this condition means

$$\begin{aligned} f_2(a_1 \otimes a_2) - g_2(a_1 \otimes a_2) &= m_1' h_2(a_1 \otimes a_2) + m_2'(f_1(a_1) \otimes h_1(a_2)) \\ &\quad + m_2'(h_1(a_1) \otimes g_1(a_2)) + h_1(m_2(a_1 \otimes a_2)) + h_2(m_1 a_1 \otimes a_2) + h_2(a_1 \otimes m_1 a_2). \end{aligned}$$

5.1.6 Bar interpretation of homotopy

The collections $\{f_i\}, \{g_i\},\ \{h_i\}$ define a homomorphism

$$D : B(M, \{m_i\}) \to B(M', \{m_i'\})$$

by

$$\begin{aligned} &D(a_1 \otimes \cdots \otimes a_n) \\ &\quad = \sum_{k_1 + \cdots + k_t = n} f_{k_1}(a_1 \otimes \cdots \otimes a_{k_1}) \otimes \cdots \otimes f_{k_{i-1}}(a_{k_1 + \cdots k_{i-2}+1} \otimes \cdots \otimes a_{k_1 + \cdots k_{i-1}}) \\ &\quad\quad \otimes h_{k_i}(a_{k_1+\cdots+k_{i-1}+1} \otimes \cdots \otimes a_{k_1+\cdots+k_i}) \otimes g_{k_{i+1}}(a_{k_1+\cdots k_i+1} \otimes \cdots \otimes a_{k_1+\cdots k_{i+1}}) \\ &\quad\quad\quad \otimes g_{k_t}(a_{k_1+\cdots k_{t-1}+1} \otimes \cdots \otimes a_{k_1+\cdots+k_t}), \end{aligned}$$

which is a $(B(\{f_i\}), B(\{g_i\}))$-coderivation.

Besides, the condition (5.4) means nothing else than

$$p_1\big(B(\{f_i\}) - B(\{g_i\})\big) = p_1(d_{m'} D + D d_m),$$

and this, again since the tensor coalgebra is cofree, gives

$$B(\{f_i\}) - B(\{g_i\}) = d_{m'} D + D d_m,$$

that is, the dg coalgebra maps $B(\{f_i\})$ and $B(\{g_i\})$ are homotopic in the category of dg coalgebras.

5.1.7 Category $DASH$

The category $DASH$ (Differential Algebras and Strongly Homotopy Multiplicative Maps) was first considered in Halperin and Stasheff's article [12]. The object are dg algebras, and a morphism $\{f_i\} : (A, d, \mu) \to (A', d', \mu')$ is defined as a collection of homomorphisms

$$\big\{f_i : \otimes^i A \to A,\ i = 1, 2, \ldots,\ \deg f_i = 1 - i\big\}$$

which satisfies the following conditions

$$\begin{aligned} &\sum_i f_n(a_1 \otimes \cdots \otimes a_{i-1} \otimes d a_i \otimes a_{i+1} \otimes \cdots \otimes a_n) \\ &\qquad + \sum_i f_{n-1}(a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i \cdot a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n) \\ &\qquad\qquad = \sum_i f_i(a_1 \otimes \cdots \otimes a_i) \cdot f_{n-i}(a_{i+1} \otimes \cdots \otimes a_n). \end{aligned} \tag{5.5}$$

In particular, for $n = 1$ this condition gives $f_1 d(a) = d f_1(a)$, i.e., $f_1 : (A, d) \to (A', d_1')$ is a chain map; for $n = 2$ it gives

$$f_1(a_1 \cdot a_2) - f_1(a_1) \cdot f_1(a_2) = d' f_2(a_1 \otimes a_2) + f_2(d a_1 \otimes a_2) + f_2(a_1 \otimes d a_2),$$

thus $f_1 : (A, d, \mu) \to (A', d', \mu')$ is multiplicative up to the homotopy f_2.

The existence of higher components $\{f_i,\ i = 1, 2, 3, 4, \dots\}$ is the reason why such a morphism is called a *Strongly Homotopy Multiplicative Map* and the category is called DASH: *Differential Algebras and Strongly Homotopy multiplicative maps.*

In fact this is a *full subcategory* of the category of A_∞-algebras whose objects are ordinary dg-algebras $(A, d, \mu) = (A, \{m_1 = d, m_2 = \mu, m_3 = 0, m_4 = 0, \dots\})$: the defining condition of an A_∞-morphism (5.2) here looks like (5.5).

Thus we have hierarchy of categories

$$DGAlg \subset DASH \subset A_\infty,$$

all three with notions of homotopies, and we have the commutative diagram of functors

$$\begin{tikzcd} DGAlg \arrow[r, phantom, "\subset"] \arrow[dr, "B"'] & DASH \arrow[r, phantom, "\subset"] \arrow[d, "B"] & A_\infty \arrow[dl, "B"] \\ & DGCoalg & \end{tikzcd}.$$

5.1.8 Minimality

Let $\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ be a morphism of A_∞-algebras. It follows from (5.2) that the first component $f_1 : (M, m_1) \to (M', m'_1)$ is a chain map.

Let us define a weak equivalence of A_∞-algebras as a morphism $\{f_i\}$ for which $B(\{f_i\})$ is a weak equivalence (homology isomorphism) of dg coalgebras. The standard spectral sequence argument allows to prove the following

Proposition 5.1. *A morphism of A_∞-algebras is a weak equivalence if and only if its first component $f_1 : (M, m_1) \to (M', m'_1)$ is a weak equivalence of chain complexes.*

Furthermore, it is easy to see that if $\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ is an isomorphism of A_∞ algebras then its first component $f_1 : (M, m_1) \to (M', m'_1)$ is an isomorphism of chaian complexes. Conversely, if f_1 is an isomorphism, then $\{f_i\}$ is an isomorphism of A_∞ algebras: The components of the opposite morphism $\{g_i\} : (M', \{m'_i\}) \to (M, \{m_i\})$ can be solved inductively from the equation

$$\{g_i\}\{f_i\} = \{id_M, 0, 0, \dots\}.$$

Thus we have the

Proposition 5.2. *A morphism of A_∞-algebras is an isomorphism if and only if its first component $f_1 : (M, m_1) \to (M', m'_1)$ is an isomorphism of dg modules.*

Definition 5.3. An A_∞-algebra $(M, \{m_i\})$ we call minimal if $m_1 = 0$.

In this case (M, m_2) is a *strictly* associative graded algebra.

The above propositions easily imply:

Proposition 5.3. *Each weak equivalence of minimal A_∞-algebras is an isomorphism.*

Proof. Suppose $f = \{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ is a weak equivalence of A_∞-algebras. Then by (5.1) the chain map $f_1 : (M, m_1 = 0) \to (M', m'_1 = 0)$ induces an isomorphism of homology modules $f_1^* : H(M, m_1 = 0) \to H(M', m'_1 = 0)$, but

$$H(M, m_1 = 0) = M \text{ and } H(M', m_1 = 0) = M'.$$

So f_1 is an isomorphism and by of (5.2) f is an isomorphism too. □

This fact motivates the word *minimal* in this notion since Sullivan's minimal model has a similar property.

5.2 A_∞ twisting cochains

Here we are going to generalize the above material about twisting cochains and the functor D from the case of dg algebras to the case of A_∞ algebras, see [18], see also [29].

Start with a dg coalgebra $(C, d_C, \nabla : C \to C \otimes C)$ and an A_∞-algebra $(M, \{m_i\})$. We'll recall the following multicooperations

$$\nabla^i : C \to C \otimes \cdots (i \text{ times}) \cdots \otimes C \to C, \quad i = 1, 2, \dots,$$
$$\nabla^1 = id, \quad \nabla^2 = \nabla, \quad \nabla^i = (\nabla^{i-1} \otimes id)\nabla.$$

5.2.1 The notion of A_∞-twisting cochain

An A_∞-twisting cochain [18], [19] we have defined as a homomorphism $\phi : C \to M$ of degree -1 (that is $\phi : C^* \to M^{*-1}$) satisfying the condition

$$\phi d_C = \sum_{i=1}^{\infty} m_i(\phi \otimes \cdots \otimes \phi)\nabla^i. \tag{5.6}$$

Let $Tw(C, M)$ be the set of all such twisting cochains.

Note that if for $(M, \{m_i\})$ one has $m_{>2} = 0$, i.e., it is an ordinary dg algebra $(M, \{m_1, m_2, 0, 0, \dots\})$ then this notion coincides with Brown' classical notion.

5.2.2 Bar interpretation

Any A_∞ twisting cochain $\phi : C \to M$ induces a dg-coalgebra map $B(\phi) : C \to BM$ by

$$B(\phi) = \sum_i (\phi \otimes \cdots \otimes \phi)\nabla^i.$$

So defined $B(\phi)$ automatically is a coalgebra map and the condition (5.6) guarantees that it is a chain map.

Conversely, any dg coalgebra map $f : C \to BM$ is $B(\phi)$ for $\phi = p \circ f : C \to BM \to M$. In fact we have a bijection

$$Mor_{DGCoalg}(C, BM) \longleftrightarrow Tw(C, M).$$

5.2.3 Equivalence of A_∞-twisting cochains

Two A_∞-twisting cochains $\phi, \psi : C \to M$ are called equivalent [18] if there exists $c : C \to M$, $\deg c = 0$, such that

$$\phi - \psi = c d_C + \sum_{i=1}^{\infty} \sum_{k=1}^{i-1} m_i\big(\phi \otimes \cdots (k \text{ times}) \cdots \phi \otimes c \otimes \psi \otimes \cdots \psi\big)\nabla^i, \tag{5.7}$$

notation $\phi \sim_c \psi$.

Again, if for $(M, \{m_i\})$ one has $m_{>2} = 0$, i.e., it is an ordinary dg algebra $(M, \{m_1, m_2, 0, 0, \dots\})$ then this notion coincides with Berikashvili's described above equivalence.

5.2.4 Bar interpretation

If $\phi \sim_c \psi$ then $B(\phi)$ and $B(\psi)$ are homotopic in the category of dg coalgebras: a chain homotopy $D(c) : C \to BA$ is given by

$$D(c) = \sum_{i=1}^{\infty} \sum_{k=1}^{i-1} (\phi \otimes \cdots (k \text{ times}) \cdots \phi \otimes c \otimes \psi \otimes \cdots \psi)\nabla^i.$$

So defined D automatically is a $B(\phi), B(\psi)$-coderivation and the condition (5.7) guarantees that it realizes chain homotopy, that is, $B(\phi) - B(\psi) = d_B D(c) + D(c) d_C$.

5.2.5 The functor $\widetilde{D}$

The functor $\widetilde{D}(C,M)$ ([18]) is defined as the factorset

$$\widetilde{D}(C,A) = \frac{Tw(C,M)}{\sim}.$$

Any A_∞-algebra map $f = \{f_i\} : M \to M'$ induces the map $Tw(C,M) \to Tw(C,M')$: if $\phi : C \to M$ is an A_∞-twisting cochain then so is

$$f\phi = p'B(f)B(\phi) = \sum_i f_i(\phi \otimes \cdots \otimes \phi)\nabla^i : C \to B(M) \to B(M') \to M'. \tag{5.8}$$

Furthermore, if $\phi \sim_c \psi$ then $f\phi \sim_C f\psi$ with

$$C = \sum_{i=1}^{\infty}\sum_{k=1}^{i-1} \big(\phi \otimes \cdots (k \text{ times}) \cdots \phi \otimes c \otimes \psi \otimes \cdots \psi\big)\nabla^i.$$

Thus we have a map $\widetilde{D}(f) : \widetilde{D}(C,M) \to \widetilde{D}(C,M')$.

Dually, any dg coalgebra map $g : C' \to C$ induces a map $Tw(C,M) \to Tw(C',M)$: if $\phi : C \to M$ is an A_∞-twisting cochain so is the composition $\phi g : C' \to C \to M$. Moreover, if $\phi \sim_c \psi$ then $\phi g \sim_{cg} \psi g$. Thus we have a map $D(g) : D(C,M) \to D(C',M)$.

5.2.6 Bar interpretation

Assigning to an A_∞-twisting cohain $\phi : C \to M$ the dg coalgebra map $B(\phi) : C \to BM$ and having in mind that $\phi \sim_c \psi$ implies $B(f\phi) \sim_{D(c)} B(\psi)$ we obtain the

Theorem 5.1. *There is a bijection $\widetilde{D}(C,M) \longleftrightarrow [C,BM]_{DGCoalg}$.*

Taking $C = B(M')$, the bar construction of an A_∞-algebra $(M',\{m'_i\})$ and having in mind that $[B(M'),B(M)]_{DGCoalg}$ is bijective to $[M',M]_{A_\infty}$, the set of homotopy classes in the category of A_∞-algebras,we obtain the

Proposition 5.4. *There is a bijection $D(B(M'),M) \longleftrightarrow [M',M]_{A_\infty}$.*

5.2.7 Bijections

The following property of the functor $\widetilde{D}$ makes it useful in the homotopy classification of maps.

Theorem 5.2 (see [18])**.**

(a) *If $\{f_i\} : M \to M'$ is a weak equivalence of A_∞-algebras, then*

$$D(\{f_i\}) : D(C,M) \to D(C,M')$$

is a bijection.

(b) *If $g : C \to C'$ is a weak equivalence of connected dg coalgebras (i.e., a homology isomorphism), then*

$$D(g) : D(C',M') \to D(C,M')$$

is a bijection.

Combining we obtain

Proposition 5.5. *A weak equivalence of A_∞-algebras $\{f_i\} : M \to M'$ and a weak equivalence of dg coalgebras $g : C \to C'$ induce bijections*

$$D(C,M) \xrightarrow{D(\{f_i\})} D(C,M') \xleftarrow{D(g)} D(C',M').$$

The following theorem is an analogue of Berikashvilis's Theorem 4.3 for A_∞-algebras and was proved in [18].

Theorem 5.3. *Let (K, d_K, ∇_K) be a dg coalgebra with free K_i and let $(M, \{m_i\})$ be a connected dg algebra. If $f = \{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ is a weak equivalence of A_∞-algebras then*

$$D_\infty(f) : D_\infty(K, M) \to D_\infty(K, M')$$

is a bijection.

In fact this theorem means that $[K, BM] \to [K, BM']$ is a bijection. The theorem consists of two parts:

Surjectivity. Any A_∞-twisting cochain $\phi : K \to M'$ can be lifted to an A_∞-twisting cochain $\psi : K \to M$ so that $\phi \sim f\psi$.

Injectivity. If $\psi \sim \psi' \in T_\infty(K, M)$, then $f\psi \sim f\psi \in T_\infty(K, M')$.

5.2.8 Lifting of A_∞-twisting cochains

Below we'll use the surjectivity part of this theorem whose proof we sketch here.

Theorem 5.4. *Let (K, d_K, ∇_K) be a dg coalgebra with free K_i and let $f : (M, \{m_i\}) \to (M', \{m'_i\})$ be a weak equivalence of connected A_∞-algebras (that is, $M_0 = M'_0 = 0$). Then for an arbitrary A_∞-twisting cochain*

$$\phi = \phi_2 + \phi_3 + \cdots + \phi_n + \cdots : K \to M'$$

there exists an A_∞-twisting cochain

$$\psi = \psi_2 + \psi_3 + \cdots + \psi_n + \cdots : K \to M$$

such that $\phi \sim f\psi$.

Proof. Start with an A_∞-twisting cochain

$$\phi = \phi_2 + \phi_3 + \cdots + \phi_n + \cdots : K \to M'.$$

The condition (5.6) gives $m_1\phi_2 = 0$, and since $f : (M, m_1) \to (M'm'_1)$ is homology isomorphism then there exist $\psi_2 : K_2 \to M_1$ and $c'_2 : K_2 \to M'_2$ such that $m_1\psi_2 = 0$ and $f\psi_2 = \phi_2 + m'_1c'_2$. Perturbing ϕ by this c'_2 we obtain $F_{c'_2}\phi$ for which $(F_{c'_2}\phi)_2 = \phi_2 + m'_1c'_2 = f\psi_2$. □

Assume now that we already have $\psi_2, \psi_3, \ldots, \psi_{n-1}$ which satisfy (5.6) and (5.8) in appropriate dimensions, that is

$$m_1\psi_k = \psi_{k-1}d_K + \sum_{i=2}^{\mathrm{int}(k/2)} \sum_{k_1+\cdots k_i=k} m_i(\psi_{k_1} \otimes \cdots \otimes \psi_{k_i})\nabla_K^k, \;\; k = 2, 3, \ldots, n-1 \tag{5.9}$$

and

$$\phi_k = \sum_{i=1}^{\mathrm{int}(k/2)} \sum_{k_1+\cdots+k_i=k} f_i(\psi_{k_1} \otimes \cdots \otimes \psi_{k_i})\nabla_K^i, \;\; k = 1, 2, \ldots, n-1. \tag{5.10}$$

We need the next component $\psi_n : K_n \to A_{n-1}$ satisfying the condition (5.9) for $k = n$

$$m_1\psi_n = \psi_{n-1}d_K + \sum_{i=2}^{\mathrm{int}(n/2)} \sum_{k_1+\cdots k_i=n} m_i(\psi_{k_1} \otimes \cdots \otimes \psi_{k_i})\nabla_K^k, \tag{5.11}$$

and $c'_n : K_n \to A'_n$ such that

$$m_1c'_n + \phi_n = \sum_{i=1}^{\mathrm{int}(n/2)} \sum_{k_1+\cdots+k_i=n} f_i(\psi_{k_1} \otimes \cdots \otimes \psi_{k_i})\nabla_K^i. \tag{5.12}$$

Then perturbing ϕ by c'_n we obtain new ϕ_n for which the condition (5.10) will be satisfied for $k=n$, and this will complete the proof.

Let us put

$$U_n = \psi_{n-1}d_K + \sum_{i=2}^{int(n/2)} \sum_{k_1+\cdots k_i=n} m_i(\psi_{k_1}\otimes\cdots\otimes\psi_{k_i})\nabla_K^k,$$

$$U'_n = \phi_{n-1}d_K + \sum_{i=2}^{int(n/2)} \sum_{k_1+\cdots k_i=n} m_i(\phi_{k_1}\otimes\cdots\otimes\phi_{k_i})\nabla_K^k,$$

$$V_n = \sum_{i=2}^{int(n/2)} \sum_{k_1+\cdots+k_i=n} f_i(\psi_{k_1}\otimes\cdots\otimes\psi_{k_i})\nabla_K^i.$$

Then the needed conditions (5.11) and (5.12) become

$$U_n = m_1\psi_n, \quad m'_1c'_n + \phi_n = f_1\psi_n + V_n.$$

First, it is possible to check that $m_1U_n = 0$ and $f_1U_n = U'_n + m'_1V_n$. Having in mind that $U'_n = m'_1\phi_n$ the last condition means $f_1U_n = m'_1(\phi_n + V_n)$. So U_n is a map to m_1-cycles

$$U_n : K_n \to Z(M_{n-2}) \subset M_{n-2}$$

and f_1U_n maps K_n to boundaries

$$f_1U_n : K_n \to B(M'_{n-2}) \subset M_{n-2}.$$

Then since $f_1 : (M, m_1) \to (M', m'_1)$ is a homology isomorphism, there exist $\overline{\psi}_n : K_n \to M_{n-1}$ such that $m_1\overline{\psi}_n = U_n$. So our $\overline{\psi}_n$ satisfies (5.11). Now let us perturb this $\overline{\phi}$ in order to catch the condition (5.12) too.

Using the above equality $f_1U_n = m'_1(\phi_n + V_n)$, we have

$$m'_1f_1\overline{\psi}_n = f_1m_1\overline{\psi}_n = f_1U_n = m'_1(\phi_n + V_n)$$

i.e., $m'_1(f_1\overline{\psi}_n - (\phi_n + V_n)) = 0$. Thus

$$z'_n = (f_1\overline{\psi}_n - (\phi_n + V_n)) : K_n \to Z(M'_{n-1}),$$

and again, since $f_1 : (M, m_1) \to (M', m'_1)$ is a homology isomorphism there exist $z_n : K_n \to Z(M_{n-1})$ and $c'_n : K_n \to M'_n$ such that $z'_n = f_1z_n - m'_1c'_n$. Let's define $\psi_n = \overline{\psi}_n - z_n$. Then

$$f_1\psi_n = f_1\overline{\psi}_n - f_1z_n = (z'_n + \phi_n + V_n) - (z'_n - m'_1c'_n) = \phi_n + V_n.$$

5.3 C_∞-algebras

This is the commutative version of the notion of A_∞-algebra. For an A_∞-algebra $(M, \{m_i\})$ it is clear what it means for the operation $m_2 : M\otimes M \to M$, but what about the commutativity of the higher operations $m_i : M\otimes\cdots\otimes M \to M,\ i \geq 3$? We are going to describe this now.

5.3.1 Shuffle product

It was mentioned in (3.2) that the tensor algebra $(T(V), \mu)$ and the tensor coalgebra $(T^c(V), \Delta)$ coincide as graded modules, but the multiplication μ of $T(V)$ and the comultiplication Δ of $T^c(V)$ are not compatible with each other, so they do not define a graded bialgebra structure on $T(V) = T^c(V)$.

But, as it was indicated in (3.2) there exists the *shuffle* multiplication $\mu_{sh} : T^c(V)\otimes T^c(V) \to T^c(V)$ introduced by Eilenberg and MacLane [7] which turns $(T^c(V), \Delta, \mu_{sh})$ into a graded bialgebra. This multiplication is defined as a graded coalgebra map induced by the universal property of $T^c(V)$ by

$\alpha : T^c(V) \otimes T^c(V) \to V$ given by $\alpha(v \otimes 1) = \alpha(1 \otimes v) = v$ and $\alpha = 0$ otherwise. This multiplication is associative and in fact is given by

$$\mu_{sh}\big([a_1, \ldots, a_m] \otimes [a_{i+1}, \ldots, a_n]\big) = \sum \pm\big([a_{\sigma(1)}, \ldots, a_{\sigma(n)}]\big),$$

where the summation is taken over all (m, n)-shuffles. That is, over all permutations of the set $(1, 2, \ldots, n+m)$ which satisfy the condition: $i < j$ if $1 \le \sigma(i) < \sigma(j) \le n$ or $n+1 \le \sigma(i) < \sigma(j) \le n+m$. In particular

$$[a] *_{sh} [b] = [a, b] \pm [b, a],$$
$$[a] *_{sh} [b, c] = [a, b, c] \pm [b, a, c] \pm [b, c, a].$$

5.3.2 The notion of C_∞-algebra

Now we can define the notion of C_∞-algebra, which is a commutative version of Stasheff's notion of A_∞-algebra.

Definition 5.4 ([10, 20, 27, 29])**.** A C_∞-algebra is an A_∞-algebra $(M, \{m_i\})$ which additionally satisfies the following condition: each operation m_i vanishes on shuffles, that is, for $a_1, \ldots, a_i \in M$ and $k = 1, 2, \ldots, i-1$

$$m_i\big(\mu_{sh}\big((a_1 \otimes \cdots \otimes a_k) \otimes (a_{k+1} \otimes \cdots \otimes a_i)\big)\big) = 0. \tag{5.13}$$

In particular, this gives

$$m_2(a \otimes b) + m_2(b \otimes a) = 0,$$
$$m_3(a \otimes b \otimes c) + m_3(a \otimes c \otimes b) + m_3(c \otimes a \otimes b) = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

i.e., the maps f_i vanish on shuffles.

Definition 5.5. A morphism of C_∞-algebras is defined as a morphism of A_∞-algebras $\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ whose components f_i vanish on shuffles, that is

$$f_i\big(\mu_{sh}(a_1 \otimes \cdots \otimes a_k) \otimes (a_{k+1} \otimes \cdots \otimes a_i)\big) = 0. \tag{5.14}$$

The composition is defined as in the A_∞-case and the bar construction argument (see (5.3.3) below) allows to show that the composition is a C_∞-morphism.

In particular, this gives

$$f_2(a \otimes b) + m_2(b \otimes a) = 0,$$
$$f_3(a \otimes b \otimes c) + m_3(a \otimes c \otimes b) + m_3(c \otimes a \otimes b) = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

i.e., the maps f_i vanish on shuffles.

In particular, for the operation m_2 we have $m_2(a \otimes b \pm b \otimes a) = 0$, so a C_∞-algebra of type $(M, \{m_1, m_2, 0, 0, \ldots\})$ is a commutative dg algebra (cdga) with the differential m_1 and strictly associative and commutative multiplication m_2. Thus the category of commutative dg algebras is a subcategory of the category of C_∞-algebras.

5.3.3 Bar interpretation of a C_∞-algebra

The notion of C_∞-algebra is motivated by the following observation. It is well known that if a dg algebra (A, d, μ) is graded commutative then the differential of the bar construction BA is not only a coderivation but also a derivation with respect to the shuffle product, so the bar construction $(BA, d_\beta, \Delta, \mu_{sh})$ of a cdga is a dg bialgebra, see (3.3.3).

By definition the bar construction of an A_∞-algebra $(M, \{m_i\})$ is a dg coalgebra

$$\widetilde{B}(M) = (T^c(s^{-1}M), d_\beta, \Delta).$$

But if $(M, \{m_i\})$ is a C_∞-algebra, then $\widetilde{B}(M)$ becomes a dg bialgebra:

Proposition 5.6. *For an A_∞-algebra $(M, \{m_i\})$ the differential of the bar construction d_β is a derivation with respect to the shuffle product if and only if each operation m_i vanishes on shuffles, that is $(M, \{m_i\})$ is a C_∞-algebra.*

Proof. The map $\Phi : T^c(s^{-1}M)\otimes T^c(s^{-1}M) \to T^c(s^{-1}M)$ defined by $\Phi = d_\beta\mu_{sh} - \mu_{sh}(d_\beta\otimes id + id\otimes d_\beta)$ is a coderivation, see the arguments in (2.2.1). Thus, according to the universal property of the tensor coalgebra (3.1.6) the map Φ is trivial if and only if $p_1\Phi = 0$, and the last condition means exactly (5.13). □

5.3.4 Bar interpretation of a morphism of C_∞-algebras

A morphism of C_∞ algebras has an analogous interpretation.

Proposition 5.7. *Let $\{f_i\} : (M, \{m_i\}) \to (M', \{m'_i\})$ be an A_∞-algebra morphism of C_∞-algebras. Then the induced map of bar constructions $\widetilde{B}\{f_i\}$ is a map of dg bialgebras if and only if each f_i vanishes on shuffles, that is $\{f_i\}$ is a morphism of C_∞-algebras.*

Proof. The map $\Psi = \widetilde{B}\{f_i\}\mu_{sh} - \mu_{sh}(\widetilde{B}\{f_i\}\otimes\widetilde{B}\{f_i\})$ is a coderivation, see the arguments in (2.2.1). Thus, according to the universal property of the tensor coalgebra (3.1.6) the map Ψ is trivial if and only if $p_1\Psi = 0$, and the last condition means exactly (5.14).

Thus the bar functor maps the subcategory of C_∞-algebras to the category of dg bialgebras. □

5.3.5 Adjunctions

The bar and cobar functors

$$B : DGAlg \to DGCoalg, \quad \Omega : DGCoalg \to DGAlg$$

are adjoint and there exist standard weak equivalences $\Omega B(A) \to A$, $C \to B\Omega C$. So $\Omega B(A) \to A$ is a *free resolution* of a *dga* A.

If A is commutative, the cobar-bar resolution is out of the category: $\Omega B(A)$ is not commutative.

In this case instead the cobar-bar functors we must use the adjoint functors Γ, $\mathcal{A}$, see [32], which we describe now.

For a commutative dg algebra the bar construction is a dg bialgebra, so the restriction of the bar construction is the functor $B : CDGAlg \to DGBialg$. Furthermore, the functor of indecomposables $Q : DGBialg \to DGLieCoalg$ maps the category of dg bialgebras to the category of dg Lie coalgebras. Let Γ be the composition

$$\Gamma : CDGAlg \xrightarrow{B} DGBialg \xrightarrow{Q} DGLieCoalg\ .$$

There is the adjoint of Γ $\mathcal{A} : DGLieCoalg \to CDGAlg$, which is dual to the Chevalley–Eilenberg functor. There is the standard weak equivalence $\mathcal{A}\Gamma A \to A$. This is the commutative analogue of the standard weak equivalence $\Omega B(A) \to A$ from (3.3.4).

6 Minimal A_∞ and C_∞-algebras and Hochschild and Harrison cohomology

Here we present the connection of the notion of minimal A_∞ (resp. C_∞) algebra with the Hochschild (resp. Harrison) cochain complexes, studied in [20], see also [25].

6.1 Hochschild cohomology

Below we describe the classical Hochschild cochain complex [14] and present also some additional structures on it which will be essential to describe minimal A_∞ algebra structures.

6.1.1 Hochschild cochain complex

Let A be an algebra and let M be a bimodule over A. The Hochschild cochain complex is defined as

$$C^*(A,M)=\sum C^n(A,M), \quad C^n(A,M)=\mathrm{Hom}(\otimes^n A,M),$$

the coboundary operator $\delta : C^n(A,M)\to C^{n+1}(A,M)$ is given by

$$\begin{aligned}&\delta f(a_1\otimes\cdots\otimes a_{n+1})\\ &\quad = a_1\cdot f(a_2\otimes\cdots\otimes a_{n+1})+\sum_k \pm f(a_1\otimes\cdots\otimes a_k\cdot a_{k+1}\otimes\cdots\otimes a_{n+1})\pm f(a_1\otimes\cdots\otimes a_n)\cdot a_{n+1}.\end{aligned}$$

The Hochschild cohomology of A with coefficients in M is defined as the homology of this cochain complex and is denoted by $Hoch^*(A,M)$.

If A and M are graded then each $C^n(A,M)$ is graded too: $C^n(A,M)=\sum_k C^{n,k}(A,M)$ where $C^{n,k}(A,M)=\mathrm{Hom}^k(\otimes^n A,M)$; here Hom^k means homomorphisms of degree k. So the Hochschild cochain complex is bigraded in this case.

It is clear that δ maps $C^{n,k}(A,M)$ to $C^{n+1,k}(A,M)$, so $(C^{*,k}(A,M),\delta)$ is a subcomplex in $(C^*(A,M),\delta)$. Thus Hochschild cohomology in this case is bigraded:

$$Hoch^n(A,M)=\sum_k Hoch^{n,k}(A,M),$$

where $Hoch^{n,k}(A,M)$ is the n-th homology module of $(C^{*,k}(A,M),\delta)$.

For our needs instead of M we take the algebra A itself. The complex $C^{*,*}(A,A)$ is a bigraded differential algebra with respect to the following cup product: for $f\in C^{m,k}(A,A)$ and $g\in C^{n,t}(A,A)$ the product $f\smile g\in C^{m+n,k+t}(A,A)$ is defined by

$$f\smile g(a_1\otimes\cdots\otimes a_{m+n})=f(a_1\otimes\cdots\otimes a_m)\cdot g(a_{n+1}\otimes\cdots\otimes a_{m+n}),$$

and the Hochschild differential is a derivative:

$$\delta(f\smile g)=\delta f\smile g\pm f\smile \delta g.$$

6.1.2 Gerstenhabers circle product

Besides this product $C^{*,*}(A,A)$ there exist much richer and important algebraic operations, finally forming a hGa (homotopy Gerstenhaber algebra) structure. First of all, there is Gerstenhaber's so called circle product [8] $f\circ g$, sometimes called Gerstenhabers brace $f\{g\}$, but let us denote it as $f\smile_1 g$ since it has properties very similar to Steenrod's $\smile_1$ product in the cochain complex of a topological space. Here is the definition of this Gerstenhabers product: for $f\in C^{m,s}(A,A)$, $g\in C^{n,t}(A,A)$ their $\smile_1$ product is defined as

$$\begin{aligned}&f\smile_1 g(a_1\otimes\cdots\otimes a_{m+n-1})\\ &\qquad=\sum_{k=0}^{m-1}\pm f\big(a_1\otimes\cdots\otimes a_k\otimes g(a_{k+1}\otimes\cdots\otimes a_{k+n})\otimes\cdots\otimes a_{m+n-1}\big)\in C^{m+n-1,s+t}(A,A).\end{aligned}$$

In [20], see also [25], it is shown that Gerstenhabers circle = brace = $\smile_1$ product satisfies Steenrod's condition

$$\delta(f\smile_1 g)=\delta f\smile_1 g\pm f\smile_1\delta g\pm f\smile g\pm g\smile f. \tag{6.1}$$

What is more amazing, $\smile_1$ also has the following property:

$$(f\smile g)\smile_1 h=f\smile(g\smile_1 h)\pm(f\smile_1 h)\smile g,$$

which means that $\cdots\smile_1 h$ is a derivation. This is an analogue of Hirsch's formula in the cochain complex of a topological space.

Besides, although the $\smile_1$ is not associative, it is possible to show that it satisfies the so called pre-Jacobi identity

$$f \smile_1 (g \smile_1 h) - (f \smile_1 g) \smile_1 h = f \smile_1 (h \smile_1 g) - (f \smile_1 h) \smile_1 g$$

which guarantees that the commutator

$$[f, g] = f \smile_1 g - g \smile_1 f$$

satisfies the Jacobi identity. Besides, the condition (6.1) implies that $[\ ,\ ] : C^{m,s}(A, A) \otimes C^{n,t}(A, A) \to C^{m+n-1,s+t}(A, A)$ is a chain map, and this implies that $(C^{*,*}(A, A), \delta, [\ ,\ ])$ is a dg Lie algebra.

By the way, the Hochschild differential δ can be expressed as

$$\delta f = f \smile_1 \mu + \mu \smile_1 f = [f, \mu]$$

where $\mu : A \otimes A \to A$ is the produduct operation of A.

6.1.3 Higher operations in the Hochschild complex

In order to perturb $\smile_1$-twisting cohains (i.e., minimal A_∞ algebra structures, as we'll see below) we need a kind of analog of the Berikishvilis equivalence of Browns twisting cochains, but now for this nonassociative $\smile_1$ case. Unfortunately for this only Gerstenhaber's $\circ = \smile_1$ is not enough, so in [20] we have introduced higher order multioperations, the $\smile_1$ products $f \smile_1 (g_1, \dots, g_i)$ of one Hochschild cohain f and a collection of Hochschild cochains $g_1, \dots, g_i$. Now these operations are called Getzler–Kadeishvili brace operations and are denoted as $f\{g_1, \dots, g_i\}$. Here is the definition:

$$\begin{aligned} &f \smile_1 (g_1, \dots, g_i)(a_1 \otimes \cdots \otimes a_{n+n_1+\cdots+n_i-i}) \\ &\quad = \sum_{k_1,\dots,k_i} f(a_1 \otimes \cdots \otimes a_{k_1} \otimes g_1(a_{k_1+1} \otimes \cdots \otimes a_{k_1+n_1}) \otimes \cdots \otimes a_{k_2} \otimes g_2(a_{k_2+1} \otimes \cdots \otimes a_{k_2+n_2}) \\ &\qquad\qquad \otimes a_{k_2+n_2+1} \otimes \cdots \otimes a_{k_i} \otimes g_i(a_{k_i+1} \otimes \cdots \otimes a_{k_i+n_i}) \otimes \cdots \otimes a_{n+n_1+\cdots+n_i-i}). \end{aligned}$$

These higher braces $f\{g_1, \dots, g_i\}$ form on the Hochschild cochain complex $C^*(A, A)$ a so called homotopy Gerstenhaber algebra (hGa) structure.

6.1.4 Hochschild cochain complex as a homotopy Gerstenhaber algebra

Let H be a graded algebra. Consider the Hochshild cochain complex of this graded algebra with coefficients in itself, $C^{*,*}(, H, H)$ which is bigraded in this case:

$$C^{n,m}(H, H) = \mathrm{Hom}^m(H^{\otimes n}, H),$$

where Hom^m means homomorphisms of degree m.

This bigraded complex carries a structure of homotopy Gerstenhaber algebra (see [10, 20, 25, 35]) $C^{*,*}(H, H), \delta, \smile, \{E_{1,k},\ k = 1, 2, \dots\}$, which consists of the following structure maps:

(i) The Hochschild differential $\delta : C^{n-1,m}(H, H) \to C^{n,m}(H, H)$ given by

$$\begin{aligned} \delta f(a_1 \otimes \cdots \otimes a_n) &= a_1 \cdot f(a_2 \otimes \cdots \otimes a_n) \\ &\quad + \sum_k \pm f(a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k \cdot a_{k+1} \otimes \cdots \otimes a_n) \pm f(a_1 \otimes \cdots \otimes a_{n-1}) \cdot a_n; \end{aligned}$$

(ii) The $\smile$ product defined by

$$f \smile g(a_1 \otimes \cdots \otimes a_{n+m}) = f(a_1 \otimes \cdots \otimes a_n) \cdot g(a_{n+1} \otimes \cdots \otimes a_{n+m}).$$

(iii) The *brace* operations $f\{g_1,\dots,g_i\}$ from [20], which we write here as

$$f\{g_1,\dots,g_i\} = E_{1,i}(f;g_1,\dots,g_i),$$
$$E_{1,i}: C^{n,m}\otimes C^{n_1,m_1}\otimes\cdots\otimes C^{n_i,m_i}\to C^{n+\sum n_t - i, m+\sum m_t},$$

given by

$$\begin{aligned}&E_{1,i}(f;g_1,\dots,g_i)(a_1\otimes\cdots\otimes a_{n+n_1+\cdots+n_i-i})\\ &=\sum_{k_1,\dots,k_i}\pm f\Big(a_1\otimes\cdots\otimes a_{k_1}\otimes g_1(a_{k_1+1}\otimes\cdots\otimes a_{k_1+n_1})\otimes\cdots\otimes a_{k_2}\otimes g_2(a_{k_2+1}\otimes\cdots\otimes a_{k_2+n_2})\\ &\qquad\otimes a_{k_2+n_2+1}\otimes\cdots\otimes a_{k_i}\otimes g_i(a_{k_i+1}\otimes\cdots\otimes a_{k_i+n_i})\otimes\cdots\otimes a_{n+n_1+\cdots+n_i-i}\Big).\end{aligned}$$

The first brace operation $E_{1,1}$ has the properties of Steenrod's $\smile_1$ product, so we use the notation $E_{1,1}(f,g) = f\smile_1 g$. In fact this is Gerstenhaber's $f\circ g$ product [8,9].

The name *Homotopy G-algebra* is motivated by the fact that this structure induces on homology $H(A)$ the structure of Gerstenhaber algebra (G-algebra).

The sequence $\{E_{1,k}\}$ defines a twisting cochain

$$E: BC^{*,*}(H,H)\otimes BC^{*,*}(H,H)\to C^{*,*}(H,H),$$

and, consequently, defines a strictly associative product on the bar construction $BC^{*,*}(H,H)$

$$\mu_E: BC^{*,*}(H,H)\otimes BC^{*,*}(H,H)\to BC^{*,*}(H,H)$$

which turns it into a DG-bialgebra.

6.2 Description of minimal A_∞-algebra structure as a twisting cochain in Hochschild complex

Here we present the connection of the notion of minimal A_∞-algebra with Hochschild cochain complexes, studied in [20], see also [25].

6.2.1 Minimal A_∞ structure as a Hochschild twisting cochain

Now let
$$\big(H,\{m_1=0,m_2,m_3,\dots,m_n,\dots\}\big)$$
be a minimal A_∞-algebra. So (H,m_2) is an associative graded algebra with multiplication $a\cdot b = m_2(a\otimes b)$ and we can consider the Hochschild cochain complex $C^{*,*}(H,H)$.

Each operation m_i can be considered as a Hochschild cochain $m_i\in C^{i,2-i}(H,H)$. Let $m = m_3+m_4+\cdots\in C^{*,2-*}(H,H)$. Stasheff's defining condition of A_∞-algebra (5.1)

$$\sum_{k=0}^{n-1}\sum_{j=1}^{n-k} m_{n-j+1}\big(a_1\otimes\cdots\otimes a_k\otimes m_j(a_{k+1}\otimes\cdots\otimes a_{k+j})\otimes\cdots\otimes a_n\big) = 0.$$

in our case looks like this:

$$\begin{aligned}&m_2\big(a_1\otimes m_{n-1}(a_2\otimes\cdots\otimes a_n)\big) + m_2\big(m_{n-1}(a_1\otimes\cdots\otimes a_{n-1})\otimes a_n\big)\\ &\qquad+\sum_k m_{n-1}\big(a_1\otimes\cdots\otimes a_k\otimes m_2(a_{k+1}\otimes a_{k+2})\otimes a_{k+3}\otimes\cdots\otimes a_n\big)\\ &\qquad\qquad=\sum_{k=0}^{n-1}\sum_{j=3}^{n-k} m_{n-j+1}\big(a_1\otimes\cdots\otimes a_k\otimes m_j(a_{k+1}\otimes\cdots\otimes a_{k+j})\otimes\cdots\otimes a_n\big).\end{aligned}$$

This can be rewritten as

$$\delta m_n = \sum_{j=3,\dots,n-2} m_{n-j+1} \smile_1 m_j.$$

And this means exactly that $\delta m = m \smile_1 m$ for $m = m_3 + m_4 + \cdots$, $m_i \in C^{i,2-i}(H,H)$.

So a minimal A_∞-algebra structure on H is, in fact, a *twisting cochain*

$$m = m_3 + m_4 + \cdots \in C^{*,2-*}(H,H)$$

in the Hochschild complex with respect to the (nonassociative) $\smile_1$ product.

6.2.2 Perturbations of minimal A_∞ algebras

Now let $(H, \{m_i\}$ be a minimal A_∞-algebra, so (H, m_2) is an associative graded algebra with multiplication $a \cdot b = m_2(a \otimes b)$.

As it was mentioned in section 6.2.1, each operation m_i can be considered as a Hochschild cochain $m_i \in C^{i,2-i}(H,H)$. Let $m = m_3 + m_4 + \cdots \in C^{*,2-*}(H,H)$. The defining condition of an A_∞-algebra (5.1) means exactly $\delta m = m \smile_1 m$. So a minimal A_∞-algebra structure on H is, in fact, a *twisting cochain* in the Hochschild complex with respect to the $\smile_1$ product.

There is the notion of equivalence of such twisting cochains: $m \sim m'$ if there exists $p = p^{2,-1} + p^{3,-2} + \cdots + p^{i,1-i} + \cdots$, $p^{i,1-i} \in C^{i,1-i}(H,H)$ such that

$$m - m' = \delta p + p \smile p + p \smile_1 m + m' \smile_1 p + E_{1,2}(m'; p, p) + E_{1,3}(m'; p, p, p) + \cdots . \tag{6.2}$$

Proposition 6.1. *Twisting cochains $m, m' \in C^{*,2-*}(H,H)$ are equivalent if and only if $(H, \{m_i\})$ and $(H', \{m'_i\})$ are isomorphic A_∞-algebras.*

Proof. Indeed,

$$\{p_i\} : (H, \{m_i\}) \to (H, \{m'_i\})$$

with $p_1 = id$, $p_i = p^{i,1-i}$ is the needed isomorphism: the condition (6.2) coincides with the defining condition (5.2) of a morphism of A_∞-algebras and the Proposition 5.2 implies that this morphism is an isomorphism. □

This gives the possibility of perturbing twisting cochains without changing their equivalence class:

Proposition 6.2. *Let m be a twisting cochain (i.e., a minimal A_∞-algebra structure on H) and let $p \in C^{n,1-n}(H,H)$ be an arbitrary cochain. Then there exists a twisting cochain $\overline{m}$, equivalent to m, such that $m_i = \overline{m}_i$ for $i \le n$ and $\overline{m}_{n+1} = m_{n+1} + \delta p$.*

Proof. The twisting cochain $\overline{m}$ can be solved inductively from the equation (6.2). □

Theorem 6.1. *Suppose for a graded algebra H that its Hochschild cohomology groups are trivial, $Hoch^{n,2-n}(H,H) = 0$ for $n \ge 3$. Then each $m \sim 0$, that is each minimal A_∞-algebra structure on H is degenerate, i.e., $(H, \{m_i\})$ is isomorphic to a (strictly associative) A_∞ algebra $(H, \{m_1 = 0, m_2, m_3 = 0, \dots\})$.*

Proof. From the equality $\delta m = m \smile_1 m$ in dimension 4 we obtain $\delta m_3 = 0$, that is, m_3 is a cocycle. Since $Hoch^{3,-1}(H,H) = 0$ there exists $p^{2,-1}$ such that $m_3 = \delta p^{2,-1}$. Perturbing our twisting cochain m by $p^{2,-1}$ we we obtain a new twisting cochain $\overline{m} = \overline{m}_3 + \overline{m}_4 + \cdots$ equivalent to m and with $\overline{m}_3 = 0$. Now the component $\overline{m}_4$ becomes a cocycle, which can be killed using $Hoch^{4,-2}(H,H) = 0$, etc. □

6.2.3 Minimal C_∞-algebra structure and Harrison cohomology

Suppose now (H, μ) is a commutative graded algebra. The Harrison cochain complex $\overline{C}^*(H,H)$ is defined as a subcomplex of the Hochschild complex consisting of cochains which disappear on shuffles. If $(H, \{m_i\})$ is a C_∞-algebra then the twisting element $m = m_3 + m_4 \cdots$ belongs to Harrison subcomplex $\overline{C}^*(H,H) \subset C^*(H,H)$ and we have the

Theorem 6.2. *Suppose for a graded commutative algebra H Harrison cohomology*

$$Harr^{n,2-n}(H,H) = 0 \ \textit{for} \ n \ge 3.$$

Then each $m \sim 0$, that is each minimal C_∞-algebra structure on H is degenerate.

7 Minimality theorem, A_∞-algebra structure in homology

7.1 A_∞-algebra structure in homology

Let (A, d, μ) be a dg algebra and $(H(A), \mu^*)$ be its homology algebra.

Although the dg algebras (A, d, μ) and $(H(A), d = 0, \mu^*)$ have same homology algebras, the homology algebra $H(A)$ carries less information that the dg algebra A.

Generally, there does not exist a map of dg algebras $H(A) \to A$ which induces an isomorphism of homology algebras. Of course, stepping from A to the smaller object $H(A)$ one looses part of the information. To compensate this loss, it is natural to enrich the algebraic structure on the smaller object $H(A)$. The classical examples of such enrichments are Steenrod squares, Massey products, ...

7.1.1 Minimality theorem

Below we present one sort of such additional algebraic structure, namely A_∞-algebra structure on the cohomology algebra, the so called minimality theorem.

It was mentioned above (5) Stasheff's A_∞-algebras are sort of Strong Homotopy Associative Algebras, the operation m_3 is a homotopy which measures the nonassociativity of the product m_2. So its existence on homology the $H(A)$, which is strictly associative looks a bit strange, but although the product on $H(A)$ is associative, there appears a structure of a (generally nondegenerate) *minimal* A_∞-algebra, which can be considered as an A_∞ *deformation* of $(H(A), \mu^*)$, [25]. Namely, in [15,16] the following *minimalty theorem* was proved:

Theorem 7.1. *Suppose that for a dg algebra (A, d, μ) all homology modules $H_i(A)$ are free.*

(a) *Then there exist a structure of minimal A_∞-algebra on $H(A)$*

$$\big(H(A), \{m_1 = 0, m_2 = \mu^*, m_3, \dots,\ m_i\}\big)$$

and a weak equivalence of A_∞-algebras

$$\{f_i\} : (H(A), \{m_i\}) \to (A, \{d, \mu, 0, 0, \dots\})$$

such that $m_1 = 0$, $m_2 = \mu^*$, $f_1^* = id_{H(A)}$.

(b) *Furthermore, for a dga map $g : A \to A'$ there exists a morphism of A_∞-algebras $\{g_i\}$: $(H(A), \{m_i\}) \to (H(A'), \{m'_i\})$ with $g_1 = g^*$ and such that the diagram*

$$\begin{tikzcd} (H(A), \{m_i\}) \arrow[r, "\{g_i\}"] \arrow[d, "\{f_i\}"'] & (H(A'), \{m'_i\}) \arrow[d, "\{f'_i\}"] \\ A \arrow[r, "g"'] & A' \end{tikzcd}$$

commutes up to homotopy in the category of A_∞ algebras.

(c) *Such a structure is unique up to isomorphism in the category of A_∞-algebras: if $(H(A), \{m_i\})$ and $(H(A), \{m'_i\})$ are two minimal A_∞-algebra structures on $H(A)$ which satisfy the conditions from (a) then there exists an isomorphism of A_∞-algebras $\{g_i\} : (H(A), \{m_i\}) \to (H(A), \{m'_i\})$ with $g_1 = id$.*

Proof. (a). We are going to construct the components f_i, m_i inductively satisfying the defining condition of an A_∞-morphism

$$\begin{aligned} df_n(a_1 \otimes \cdots \otimes a_n) &= f_1 m_n(a_1 \otimes \cdots \otimes a_n) \\ &\quad + \sum_{j=2}^{n-1} \sum_{k=0}^{n-j} f_{n-j+1}(a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes \cdots \otimes a_n) \\ &\quad + \sum_{k=1}^{n-1} f_k(a_1 \otimes \cdots \otimes a_k) \cdot f_{n-k}(a_{k+1} \otimes \cdots \otimes a_n). \end{aligned} \tag{7.1}$$

Let us start with a cycle-choosing homomorphism $f_1 : H(A) \to A$, that is $f_1(a) \in a \in H(A)$ which can be constructed using the freeness of the modules $H_i(A)$. This map is not multiplicative but $f_1(a_1 \cdot a_2) - f_1(a_1) \cdot f(a_2) \sim 0 \in A$. So there exists $f_2 : H(A) \otimes H(A) \to A$ s.t. $f_1(a_1 \cdot a_2) - f_1(a_1) \cdot f(a_2) = df_2(a_1 \otimes_2)$, and this, assuming $m_2(a_1 \otimes a_2) = a_1 \cdot a_2$, is exactly the condition (7.1) for $n = 2$.

Let us denote

$$U_3(a_1 \otimes a_2 \otimes a_3) = f_1(a_1) \cdot f_2(a_2 \otimes a_3) + f_2(a_1 \cdot a_2 \otimes a_3) + f_2(a_1 \otimes a_2 \cdot a_3) + f_2(a_1 \otimes a_2) \cdot f_1(a_3).$$

Note that the main condition (7.1) for $n = 3$ looks as follows

$$df_3(a_1 \otimes a_2 \otimes a_3) = f_1 m_3(a_1 \otimes a_2 \otimes a_3) - U_3(a_1 \otimes a_2 \otimes a_3).$$

Direct calculation shows that $dU_3(a_1 \otimes a_2 \otimes a_3) = 0$, so $U_3(a_1 \otimes a_2 \otimes a_3)$ is a cycle. Let us define $m_3(a \otimes b \otimes c) \in H(A)$ as the homology class of this cycle

$$\begin{aligned} m_3(a \otimes b \otimes c) &= \{U_3(a \otimes b \otimes c)\} \\ &= f_1(a_1) \cdot f_2(a_2 \otimes a_3) + f_2(a_1 \cdot a_2 \otimes a_3) + f_2(a_1 \otimes a_2 \cdot a_3) + f_2(a_1 \otimes a_2) \cdot f_1(a_3). \end{aligned}$$

Then, since f_1 is a *cycle choosing* homomorphism, $f_1 m_3(a_1 \otimes a_2 \otimes a_3) - U_3(a_1 \otimes a_2 \otimes a_3)$ is homological to zero. Thus, again using the freeness of $H_i(A)$, it is possible to construct a homomorphism $f_3 : H(A) \otimes H(A) \otimes H(A) \to A$ such that

$$df_3(a_1 \otimes a_2 \otimes a_3) = f_1 m_3(a_1 \otimes a_2 \otimes a_3) - U_3(a_1 \otimes a_2 \otimes a_3),$$

and this is exactly the condition (7.1) for $n = 3$.

Assume now that $f_i,\ m_i,\ i \le n-1$ are already constructed and they satisfy (7.1).

Let us denote

$$\begin{aligned} U_n(a_1 \otimes \cdots \otimes a_n) = \sum_{j=2}^{n-1} \sum_{k=0}^{n-j} & f_{n-j+1}(a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes \cdots \otimes a_n) \\ &+ \sum_{k=1}^{n-1} f_k(a_1 \otimes \cdots \otimes a_k) \cdot f_{n-k}(a_{k+1} \otimes \cdots \otimes a_n). \end{aligned}$$

Then the defining condition (7.1) can be rewritten as

$$df_n(a_1 \otimes \cdots \otimes a_n) = f_1 m_n(a_1 \otimes \cdots \otimes a_n) + U_n(a_1 \otimes \cdots \otimes a_n). \tag{7.2}$$

Direct calculation shows that $dU_n = 0$, so we define $m_n(a_1 \otimes \cdots \otimes a_n)$ as the homology class of the cycle $U_n(a_1 \otimes \cdots \otimes a_n)$. Then using the freeness of modules the $H_i(A)$ one can construct a homomorphism $f_n : H_*(A) \otimes \cdots \otimes H(A) \to A$ satisfying (7.2), and this completes the proof of (a).

(b). The morphism of A_∞-algebras $(H(A), \{m_i\}) \xrightarrow{\{f_i\}} A \xrightarrow{g} A'$ induces a twisting cochain $\phi : B(H(A), \{m_i\}) \to A$.

Furthermore, since

$$(H(A'), \{m_i'\}) \xrightarrow{\{f_i'\}} A'$$

is a weak equivalence of A_∞-algebras, by the lifting theorem (5.4) ϕ can be lifted to the A_∞-twisting cochain $\psi : B(H(A), \{m_i\}) \to (H(A'), \{m_i'\})$ which in fact represents an A_∞-morphism $\{g_i\} : (H(A), \{m_i\}) \to (H(A'), \{m_i'\})$, such that $\{f_i'\} \circ \psi \sim \phi$. This equivalence means that A_∞-morphisms $\{f_i'\} \circ \{g_i\}$ and $g \circ \{g_i\}$ are homotopic. This completes the proof.

(c). Suppose for a dg algebra (A, d, μ) we have two homology A_∞-algebras $(H(A), \{m_i\})$ and $(H(A), \{m_i'\})$, i.e., there exist A_∞ weak equivalences

$$\{f_i\} : (H(A), \{m_i\}) \to A, \quad \{f_i'\} : (H(A), \{m_i\}) \to A$$

with $(f_1)^* = (f_1')^* = id_{H(A)}$. Then, using the part (b) for $d = id_A$ we obtain

$$\{g_i\} : (H(A), \{m_i\}) \to (H(A), \{m_i'\})$$

with $g_1 = g^* = id_{H(A)}$. So, $\{g_i\}$ is an isomorphism of A_∞ algebras by (5.1). $\square$

Corollary 7.1. *The mapping of differential coalgebras*

$$B(\{f_i\}) : \widetilde{B}(H(A), \{m_i\}) \to B(A)$$

induces an isomorphism in homology.

7.1.2 Connection with Massey products

The first new operation $m_3(a_1 \otimes a_2 \otimes a_3) \in H(A)$ was defined as the homology class of the cycle

$$U_3(a_1 \otimes a_2 \otimes a_3) = f_1(a_1) \cdot f_2(a_2 \otimes a_3) + f_2(a_1 \cdot a_2 \otimes a_3) + f_2(a_1 \otimes a_2 \cdot a_3) + f_2(a_1 \otimes a_2) \cdot f_1(a_3).$$

This description immediately induces the connection of m_3 with the Massey product: If $a, b, c \in H(A)$ is a Massey triple, i.e., if $a_1 \cdot a_2 = a_2 \cdot a_3 = 0$, then

$$U_3(a_1 \otimes a_2 \otimes a_3) = f_1(a_1) \cdot f_2(a_2 \otimes a_3) + f_2(a_1 \otimes b_2) \cdot f_1(a_3)$$

and this is exactly the combination which defines the Massey product $\langle a_1, a_2, a_3\rangle$, thus $m_3(a_1 \otimes a_2 \otimes a_3) \in \langle a_1, a_2, a_3\rangle$. This gives examples of dg algebras with essentially nontrivial homology A_∞-algebras.

7.1.3 Special cases

Taking $(A, d, \mu) = C^*(X)$, the cochain complex of a topological space X, the theorem defines on the cohomology algebra $H^*(X)$ the structure of a minimal A_∞-algebra $(H^*(X), \{m_i\})$, which carries more information about X than just the graded algebra structure. In particular, we shall show later that this cohomology A_∞-algebra $(H^*(X), \{m_i\})$ determines the cohomology modules of the loop space ΩX whereas the bare the cohomology algebra $(H^*(X), m_2)$ does not.

Furthermore, taking $(A, d, \mu) = C_*(G)$, the chain complex of a topological group, the theorem defines on the Pontriagin ring $H_*(G)$ the structure of a minimal A_∞-algebra $(H_*(G), \{m_i\})$ which will carry more information about G than just the ring structure. In particular, we shall show later that this homology A_∞-algebra $(H_*(G), \{m_i\})$ determines homology modules of classifying space B_G whereas the bare Pontriagin ring $(H_*(G), m_2)$ does not.

7.1.4 Minimal A_∞-algebra structure on the homology of an A_∞-algebra

The Minimality Theorem is true also when, instead of a dg algebra (A, d, μ) we take an arbitrary $A(\infty)$-algebra $(M, \{m_i\})$.

In this case, minimal A_∞-algebra structure appears on the homology $H(M)$ of the dg module $(M, m_1 : M \to M)$, see [17].

This structure had applications in string theory, see for example [1].

8 Applications of the minimality theorem

8.1 Application: cohomology A_∞ algebra of a space and cohomology modules of loop space

Taking $A = C^*(X)$, the cochain dg algebra of a 1-connected space X, we obtain an A_∞-algebra structure $(H^*(X), \{m_i\})$ on the cohomology algebra $H^*(X)$.

The cohomology algebra equipped with this additional structure carries more information than just the cohomology algebra. Some applications of this structure are given in [16, 22]. For example the cohomology A_∞-algebra $(H^*(X), \{m_i\})$ determines the cohomology of the loop space $H^*(\Omega X)$ whereas the bare algebra $(H^*(X), m_2)$ does not:

Theorem 8.1. $H(B(H^*(X), \{m_i\})) = H^*(\Omega X)$.

Firstly, according to the minimality theorem the homology of the dg algebra $(C^*(X), d, \smile)$ carries a structure of minimal A_∞-algebra $(H^*(X), \{m_i\})$, such that there is and a weak equivalence of A_∞-algebras

$$\{f_i\} : (H^*(X), \{m_i\}) \to \big(C^*(X), \{m_1 = d, m_2 = \smile, m_3 = 0, \dots\}\big),$$

which induces weak equivalences of their bar constructions and an isomorphism of graded modules

$$H\big(B(H^*(X), \{m_i\})\big) \approx H(BC^*(X)) \approx H^*(\Omega X).$$

Thus the object $(H^*(X), \{m_i\})$, which is called the cohomology A_∞-algebra of X, determines the cohomology *modules* of the loop space $H^*(\Omega X)$. But not the cohomology *algebra* $H^*(\Omega X)$.

8.2 Application: homology modules of the classifying space of a topological group

Taking $A = C_*(G)$, the chain dg algebra of a topological group G, we obtain an A_∞-algebra structure $(H_*(G), \{m_i\})$ on the Pontriagin algebra $H_*(G)$. The homology A_∞-algebra $(H_*(G), \{m_i\})$ determines the homology of the classifying space $H_*(B_G)$ when just the Pontryagin algebra $(H_*(G), m_2)$ does not:

Theorem 8.2. *$H(B(H_*(G), \{m_i\})) = H_*(B_G)$.*

8.3 Application: A_∞-model of a fibre bundle

The minimality theorem (7.1) and the theorem (5.4) about lifting of twisting cochains allow to construct effective models of fibre bundles. Actually this model and higher operations $\{m_i\}$ and $\{p_i\}$ were constructed in [15]. Later we have recognized that they form Stasheff's A_∞-structures and the model in these terms was presented in [16]. A similar model was also presented in [29].

Topological level. Let $\xi = (X, p, B, G)$ be a principal G-fibration. Let F be a G-space. Then the action $G \,\mathrm{times}\, F \to F$ determines the *associated fibre bundle* $\xi(F) = (E, p, B, F, G)$ with fiber F. Thus ξ and the action $G \,\mathrm{times}\, F \to F$ on the topological level determine E.

Chain level. Let $K = C_*(B)$, $A = C_*(G)$, $P = C_*(F)$. The classical result of E. Brown [5] states that the principal fibration ξ determines a twisting cochain $\phi : K = C_*(B) \to A = C_*(G)$, and the action on the chain level $C_*(G) \otimes C_*(F) \to C_*(F)$ defines the twisted tensor product $K \otimes_\phi P = C_*(B) \otimes_\phi C_*(F)$ which gives the homology modules of the total space $H_*(E)$. Thus ξ and the action on the chain level $C_*(G) \otimes C_*(F) \to C_*(F)$ determine $H_*(E)$.

The twisting cochain ϕ is not uniquely determined and it can be perturbed by the above equivalence relations for computational reasons.

Homology level. Nodar Berikashvili stated the problem to lift the previous "chain level" model of associated fibration to "homology level", i.e., to construct a "twisted differential" on $C_*(B) \otimes H_*(F)$. Investigation had shown that the principal fibration ξ and the action of of Pontryagin ring $H_*(G)$ on $H_*(F)$, that is the pairing $H_*(G) \otimes H_*(F) \to H_*(F)$ *do not determine* $H_*(E)$. But by the minimality theorem it appeared that $H_*(G)$ carries not only the Pontryagin product $H_*(G) \otimes H_*(G) \to H_*(G)$ but also richer algebraic structure, namely the structure of a minimal A_∞-algebra $(H_*(G), \{m_i\})$.

Furthermore, the action $G \,\mathrm{times}\, F \to F$ induces not only the pairing $H_*(G) \otimes H_*(F) \to H_*(F)$ but, by the modular analog of the Minimality Theorem [16] also the structure of minimal A_∞-module $(H_*(F), \{p_i\})$,

$$p_i : H_*(G) \otimes \cdots ((i-1) \ \text{times}) \cdots \otimes H_*(G) \otimes H_*(F) \to H_*(F),$$

and all these operations allow to define the correct differential on $C_*(B) \otimes H_*(F)$: there is a weak equivalence, homology isomorphism

$$C_*(B) \otimes_\psi H_*(F) = K \otimes_\psi H(P) \to K \otimes_\phi P = C_*(B) \otimes_\phi C_*(F) \sim C_*(E).$$

9 C_∞-algebra structure in the homology of a commutative dg algebra, applications in Rational Homotopy Theory

There is a commutative version of the above Minimality Theorem, see [21, 22, 27]:

Theorem 9.1. *Suppose that for a commutative dg algebra A all homology R-modules $H^i(A)$ are free.*

Then there exist: a structure of minimal C_∞-algebra $(H(A), \{m_i\})$ on $H(A)$ and a weak equivalence of C_∞-algebras

$$\{f_i\} : (H(A), \{m_i\}) \to (A, \{d, \mu, 0, 0, \dots\})$$

such that $m_1 = 0$, $m_2 = \mu^*$, $f_1^* = id_{H(A)}$.

Furthermore, for a cdga map $f : A \to A'$ there exists a morphism of C_∞-algebras

$$\{f_i\} : (H(A)\{m_i\}) \to (H(A')\{m'_i\})$$

with $f_1 = f^*$.

Such a structure is unique up to isomorphism in the category of C_∞-algebras.

Below we present some applications of this C_∞-algebra structure in rational homotopy theory.

9.1 Classification of rational homotopy types

Let X be a 1-connected space. In the case of rational coefficients there exist Sullivan's *commutative* cochain complex $A(X)$ of X. It is well known that the weak equivalence type of a cdg algebra $A(X)$ determines the rational homotopy type of X: 1-connected X and Y are *rationally homotopy equivalent* if and only if $A(X)$ and $A(Y)$ are weakly equivalent cdg algebras. Indeed, in this case $A(X)$ and $A(Y)$ have *isomorphic* minimal models $M_X \approx M_Y$, and this implies that X and Y are rationally homotopy equivalent. This is the key geometrical result of Sullivan which we are going to exploit below.

Now we take $A = A(X)$ and apply Theorem 9.1. Then we obtain on $H(A) = H^*(X, Q)$ a structure of minimal C_∞ algebra $(H^*(X, Q), \{m_i\})$ which we call the *rational cohomology C_∞-algebra of X*.

Generally, an isomorphism of rational cohomology algebras $H^*(X, Q)$ and $H^*(Y, Q)$ does not imply a homotopy equivalence $X \sim Y$, not even rationally. We claim that $(H^*(X, Q), \{m_i\})$ is a *complete* rational homotopy invariant:

Theorem 9.2. *1-connected X and X' are rationally homotopy equivalent if and only if*

$$(H^*(X, Q), \{m_i\}) \text{ and } (H^*(X', Q), \{m'_i\})$$

are isomorphic as C_∞-algebras.

Proof. Suppose $X \sim X'$. Then $A(X)$ and $A(X')$ are weakly equivalent, that is there exists a cgda A and weak equivalences $A(X) \leftarrow A \to A(X')$. This implies weak equivalences of the corresponding homology C_∞-algebras

$$(H^*(X, Q), \{m_i\}) \leftarrow (H^*(A), \{m_i\}) \to (H^*(X', Q), \{m'_i\}),$$

which by of minimality are both isomorphisms.

Conversely, suppose

$$(H^*(X, Q), \{m_i\}) \approx (H^*(X', Q), \{m'_i\}).$$

Then

$$\mathcal{A}QB(H^*(X, Q), \{m_i\}) \approx \mathcal{A}QB(H^*(X', Q), \{m'_i\}).$$

Denote this cdga as A. Then we have weak equivalences of CGD algebras (see (5.3.5))

$$A(X) \leftarrow \mathcal{A}\Gamma A(X) \leftarrow A \to \mathcal{A}\Gamma A(X') \to A(X'). \qquad \square$$

This theorem in fact classifies rational homotopy types with given cohomology algebra H as all possible minimal C_∞-algebra structures on H modulo C_∞ isomorphisms.

Example 9.1. Here we describe an example which we will use to illustrate the results of this and forthcoming sections.

We consider the following commutative graded algebra. Its underlying graded Q-vector space has the following generators: a generator e of dimension 0, generators x, y of dimension 2, and a generator z of dimension 5, so

$$H^* = \{H^0 = Q_e, 0, H^2 = Q_x \oplus Q_y, 0, 0, H^5 = Q_z, 0, 0, \dots\},$$

and the multiplication is trivial by dimensional reasons, with unit e. In fact

$$H^* = H^*(S^2 \vee S^2 \vee S^5, Q).$$

This example was considered in [12]. It was shown there that there are just two rational homotopy types with such cohomology algebra.

The same result can be obtained from our classification.

What minimal C_∞-algebra structures are possible on H^*?

For dimensional reasons only one nontrivial operation $m_3 : H^2 \otimes H^2 \otimes H^2 \to H^5$ is possible.

The specific condition of a C_∞-algebra, namely the disappearance on shuffles implies that

$$m_3(x,x,x) = 0, \ \ m_3(y,y,y) = 0, \ \ m_3(x,y,x) = 0, \ \ m_3(y,x,y) = 0$$

and

$$m_3(x,x,y) = m_3(y,x,x), \ \ m_3(x,y,y) = m_3(y,y,x).$$

Thus each C_∞-algebra structure on H^* is characterized by a pair of rational numbers p, q and

$$m_3(x,x,y) = pz, \ \ m_3(x,y,y) = qz.$$

So let us write an arbitrary minimal C_∞-algebra structure on H^* as a column vector $\begin{pmatrix} p \\ q \end{pmatrix}$.

Now let us look at the structure of an isomorphism of C_∞-algebras

$$\{f_i\} : (H^*, m_3) \to (H^*, m_3').$$

Again for dimensional reasons just one component $f_1 : H^* \to H^*$ is possible, which in its turn consists of two isomorphisms

$$f_1^2 : H^2 = Q_x \oplus Q_y \to H^2 = Q_x \oplus Q_y, \ \ f_1^5 : H^5 = Q_z \to H^5 = Q_z.$$

The first one is represented by a nondegenerate matrix $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$,

$$f_1^2(x) = ax \oplus by, \quad f_1^2(y) = cx \oplus dy,$$

and the second one by a nonzero rational number r, $f_1^5(z) = rz$.

A calculation shows that the condition $f_1^5 m_3 = m_3'(f_1^2 \otimes f_1^2 \otimes f_1^2)$, to which the defining condition of an A_∞-algebra morphism (5.2)degenerates looks as follows

$$r \begin{pmatrix} p \\ q \end{pmatrix} = \det A \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix}.$$

This condition shows that two minimal C_∞-algebra structures $m_3 = \begin{pmatrix} p \\ q \end{pmatrix}$ and $m_3' = \begin{pmatrix} p' \\ q' \end{pmatrix}$ are isomorphic if and only if they are related by a nondegenerate linear transformation.

Thus there exist just two isomorphism classes of minimal C_∞-algebras on H^*: the trivial one $(H^*, m_3 = 0)$ and the nontrivial one $(H^*, m_3 \neq 0)$. So we have just two rational homotopy types whose rational cohomology is H^*. We denote them by X and Y, respectively and analyze them in the next sections.

Below we give some applications of the cohomology C_∞-algebra in various problems of rational homotopy theory.

9.2 Formality

Among rational homotopy types with given cohomology algebra, there is one called *formal* which is a "formal consequence of its cohomology algebra" (Sullivan). Explicitly, this is the type whose minimal model M_X is isomorphic to the minimal model of the cohomology $H^*(X,Q)$.

Our C_∞ model implies the following criterion of formality:

Theorem 9.3. *X is formal if and only if its cohomology C_∞-algebra is degenerate, i.e., it is C_∞ isomorphic to one with $m_{\geq 3} = 0$.*

Below we deduce some known results about formality using this criterion.

1. A commutative graded 1-connected algebra H is called *intrinsically formal* if there is only one homotopy type with this cohomology algebra H, of course the formal one.

The above Theorem 6.2 immediately implies the following sufficient condition for formality due to Tanre [33]:

Theorem 9.4. *If for a 1-connected graded Q-algebra H one has*

$$Harr^{k,k-2}(H,H) = 0, \quad k = 3,4,\dots,$$

then H is intrinsically formal, that is there exists only one rational homotopy type with $H^*(X,Q) \approx H$.

2. The following theorem of Halperin and Stasheff from [12] is an immediate result of our criterion:

Theorem 9.5. *A commutative graded Q-algebra of type*

$$H = \left\{H^0 = Q, 0, 0, \dots, 0, H^n, H^{n+1}, \dots, H^{3n-2}, 0, 0, \dots\right\}$$

is intrinsically formal

Proof. Since $\deg m_i = 2 - i$ there is no room for operations $m_{i>2}$, indeed the shortest range is

$$m_3 : H^n \otimes H^n \otimes H^n \to H^{3n-1} = 0.$$

□

3. Theorem 9.3 easily implies the

Theorem 9.6. *Any 1-connected commutative graded algebra H with $H^{2k} = 0$ for all k is intrinsically formal.*

Proof. Any A_∞-operation m_i has degree $2 - i$, thus

$$m_i : H^{2k_1+1} \otimes \cdots \otimes H^{2k_i+1} \to H^{2(k_1+\cdots+k_i+1)} = 0.$$

Thus any C_∞-operation is trivial too. □

This implies a result of Baues: any space whose even-dimensional cohomologies are trivial has the rational homotopy type of a wedge of spheres. Indeed, such an algebra is realized by wedge of spheres and because of intrinsical formality this is the only homotopy type.

Example 9.2. Example. The algebra H^* from the example of the previous section is not intrinsically formal since there are two homotopy types, X and Y, with $H^*(X,Q) = H^* = H^*(Y)$. The space X is formal (and actually $X = S^2 \vee S^2 \vee S^5$), since its cohomology C_∞-algebra $(H^*, m_3 = 0)$ is trivial. But the space Y is not: its cohomology C_∞-algebra $(H^*, m_3 \neq 0)$ is not degenerate.

We remark here that the formal type is represented by $X = S^2 \vee S^2 \vee S^5$ and it is possible to show that the nonformal one is represented by $Y = S^2 \vee S^2 \bigcup_{f:S^4 \to S^2 \vee S^2} e^5$, where the attaching map f is a nontrivial element from $\pi_4(S^2 \vee S^2) \otimes Q$.

9.3 Rational homotopy groups

Since the cohomology C_∞-algebra $(H^*(X,Q),\{m_i\})$ determines the rational homotopy type it must determine the rational homotopy groups $\pi_i(X)\otimes Q$ too. We present a chain complex whose homology is $\pi_i(X)\otimes Q$. Moreover, the Lie algebra structure is determined as well.

For a cohomology C_∞-algebra $(H^*(X,Q),\{m_i\})$ the bar construction $B(H^*(X,Q),\{m_i\})$ is a dg bialgebra, see (5.3.3). Acting on this bialgebra by the functor Q of indecomposables we obtain a dg Lie coalgebra.

On the other hand the rational homotopy groups $\pi_*(\Omega X)\otimes Q$ form a graded Lie algebra with respect to Whiethead product. Thus its dual cohomotopy groups $\pi^*(\Omega X,Q)=(\pi_*(\Omega X)\otimes Q)^*$ form a graded Lie coalgebra.

Theorem 9.7. *Homology of a dg Lie coalgebra $QB(H^*(X,Q),\{m_i\})$ is isomorphic to the cohomotopy Lie coalgebra $\pi^*(\Omega X,Q)$.*

Proof. The theorem follows from the sequence of graded Lie coalgebra isomorphisms:

$$\begin{aligned}\pi^*(\Omega X,Q)&\approx(\pi_*(\Omega X,Q))^*\approx(PH_*(\Omega x,Q)^*\approx QH^*(\Omega X,Q)\\&\approx QH(B(A(X))\approx QH(\widetilde{B}(H^*(X,Q),\{m_i\})\approx H\big(Q\widetilde{B}(H^*(X,Q),\{m_i\})\big).\quad\square\end{aligned}$$

Example 9.3. For the algebra H^* from the previous examples the complex $QB(H^*)$ in low dimensions looks as

$$0\longrightarrow Q_x\oplus Q_y\xrightarrow{0}Q_{x\otimes x}\oplus Q_{x\otimes y}\oplus Q_{y\otimes y}\xrightarrow{0}Q_{x\otimes x\otimes y}\oplus Q_{x\otimes y\otimes y}\xrightarrow{d=m_3}Q_z\oplus\cdots.$$

The differential $d=m_3$ is trivial for the formal space X and is nontrivial for Y. Thus for both rational homotopy types we have

$$\pi^2=H^1(QB(H^*))=2Q,\quad \pi^3=H^2(QB(H^*))=3Q,$$

and

$$\begin{aligned}\pi^4(X)&=H^3(QB(H^*),d=0)=2Q,\\ \pi^4(Y)&=H^3(QB(H^*),d\neq 0)=\operatorname{Ker}d=Q.\end{aligned}$$

9.4 Realization of homomorphisms

Let $G:H^*(X,Q)\to H^*(Y,Q)$ be a homomorphism of cohomology algebras. When this homomorphism is realizable as a map of rationalizations $g:Y_Q\to X_Q$, $g^*=G$? In the case when G is an isomorphism this question was considered in [12]. It was considered also in [34]. The following theorem gives the complete answer:

Theorem 9.8. *A homomorphism G is realizable if and only if it is extendable to a C_∞-map*

$$\{g_1=G,g_2,g_3,\dots\}:(H^*(X,Q),\{m_i\})\to(H^*(Y,Q),\{m_i'\}).$$

Proof. One direction of this is consequence of the last part of Theorem 9.1.

To show the other direction we use Sullivan's minimal models M_X and M_Y of $A(X)$ and $A(Y)$. It is enough to show that the existence of $\{g_i\}$ implies the existence of cdg algebra map $g:M_Y\to M_X$.

So we have C_∞-algebra maps

$$M_X\xleftarrow{\{f_i\}}(H^*(X,Q),\{m_i\})\xrightarrow{\{g_i\}}(H^*(y,Q),\{m_i'\})\xrightarrow{\{f_i'\}}M_Y.$$

Recall the following property of a minimal cdg algebra M: for a weak equivalence of cdg algebras $\phi:A\to B$ and a cdg algebra map $f:M\to B$ there exists a cdg algebra map $F:M\to A$ such that ϕF is homotopic to f. Using this property it is easy to show the existence of a cdga map $\beta:M_X\to\mathcal{A}QB(M_X)$, the right inverse of the standard map $\alpha:\mathcal{A}QB(M_X)\to M$. Composing this map with $\mathcal{A}QB(\{f_i'\})\mathcal{A}QB(\{g_i\})$ we obtain a cdga map

$$\mathcal{A}QB(\{f_i'\})\mathcal{A}QB(\{g_i\})\beta:M_X\to M_Y.\qquad\square$$

This theorem immediately implies the

Corollary 9.1. *For formal X and Y each $G : H^*(X,Q) \to H^*(Y,Q)$ is realizable.*

Proof. In this case $\{G, 0, 0, \dots\}$ is a C_∞-extension f G. □

Example 9.4. Consider the homomorphism

$$G : H^*(X) = H^*(Y) \to H^*(S^5)$$

induced by the standard imbedding $g : S^5 \to X = S^2 \vee S^2 \vee S^5$. Of course G is realizable as $g : S^5 \to X$ but not as $S^5 \to Y$. Indeed, for such realizability, according to Theorem 9.8, we need a C_∞-algebra morphism

$$\{g_i\} : (H^*, \{0, 0, m_3, 0, \dots\}) \to (H^5(S^5, Q), \{0, 0, 0, \dots\})$$

with $g_1 = G$. For dimensional reasons all the components $g_2, g_3, \dots$ all are trivial, so this morphism has the form $\{G, 0, 0, \dots\}$. But this collection is not a morphism of C_∞-algebras since the condition $Gm_3 = 0$, to which the defining condition (5.2) of an A_∞-algebra morphism degenerates, is not satisfied.

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Torwads to Noncommutative Dynamical Systems

Andrey Krutov

Abstract

We review the classical theory of Hamiltonian dynamical systems and discuss their possible noncommutative analogues.

1 Introduction

This note follows the series of lectures given by the author at the summer school *"Algebra, Topology and Analysis: C^* and A_∞ Algebras"* in Georgia. In § 2, we recall the basic facts about symplectic manifolds and Hamiltonian dynamical systems. In particular, we recall the celebrated Liouville's theorem. In § 3, we discuss examples of noncommutative symplectic manifolds, namely, the irreducible quantum flag manifolds and discuss a possible noncommutative analogue of Hamiltonian dynamical systems.

2 Hamiltonian dynamical systems

In this section we recall the notion of classical Hamiltonian systems. We refer the reader for more details to the seminal book [4].

Let M be a smooth n-dimensional manifold. To every vector field X on M we associate the one-parametric group g^t_X of diffeomorphisms of M or *flow of* X for which X is the velocity vector field:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g^t_X(x) = X_x \quad \text{for } x \in M.$$

The two flows g^t_X and g^s_Y commute if and only if the commutator of the corresponding vector fields $[X, Y]$ is equal to zero.

A *dynamical system* is a smooth vector field X on a manifold M. In local coordinates $x_1, \ldots, x_n$, one can write

$$X = \xi_1(x_1, \ldots, x_n)\,\frac{\partial}{\partial x_1} + \cdots + \xi_n(x_1, \ldots, x_n)\,\frac{\partial}{\partial x_n}$$

and the corresponding dynamical system will have the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\,x_i = \xi_i(x_1, \ldots, x_n) \quad \text{for } i = 1, \ldots, n. \tag{2.1}$$

The flow of X corresponds to the solutions of (2.1).

A *first integral* of a dynamical system (M, X) is a (non-constant) smooth function F on M which is constant along the flow of X.

2.1 Symplectic manifolds

Let M be an even-dimensional manifold. A *symplectic structure* on M is a closed nondegenerate differential 2-form ω on M:

$$\mathrm{d}\omega = 0, \qquad \underbrace{\omega \wedge \cdots \wedge \omega}_{(\dim M)\text{ times}} \neq 0.$$

The pair (M, ω) is called a *symplectic manifolds*.

For example, consider $M = \mathbb{R}^{2n}$ with coordinates p_i, q_i and let $\omega = \sum\limits_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i$, then (M, ω) is a symplectic manifold.

2.1.1 The cotangent bundle and its symplectic structure

Let N be an n-dimensional manifold. A 1-form on the tangent space T_xN at $x \in N$ is called a *cotangent vector* to N at x. Denote the space of cotangent vectors by T_x^*V. The union of the cotangent spaces T_x^*M at all $x \in M$ is called the *cotangent bundle* of N. The cotangent bundle has a natural structure of a vector bundle over M. Hence, T^*V is a manifold. If $q = (q_1, \dots, q_n)$ is a choice of local coordinates on N, then it induces a choice of coordinates $p = (p_1, \dots, p_1)$ on the fibres of T^*N. The cotangent bundle T^*N has a natural symplectic structure given in the local coordinates by

$$\omega = \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \cdots + \mathrm{d}p_n \wedge \mathrm{d}q_n.$$

2.1.2 Hamiltonian vector fields

Let (M, ω) be a symplectic manifold. To each vector $X \in T_xM$ at $x \in M$ we can associate a 1-form ω_X on T_xM by the formula

$$\omega_X(Y) := \omega(Y, X) \quad \text{for all} \quad Y \in T_xM.$$

In fact, the linear map $X \mapsto \omega_X$ defines an isomorphism between T_xM and T_x^*M. We will denote this isomorphism by $I\colon T^*M \to TM$. The vector field $X_H := I(\mathrm{d}H)$ is called a *Hamiltonian vector field*; H is called the *Hamiltonian function* (or just *Hamiltonian*).

For $H \in C^\infty(M)$, the vector field X_H defines the corresponding 1-parameter group of diffeomorphisms $g_H^t\colon M \to M$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g_H^t(x) = X_H(x).$$

The group g_H^t is called the *Hamiltonian phase flow with Hamiltonian H*. A Hamiltonian phase flow preserves the symplectic form

$$(g_H^t)^*\omega = \omega.$$

2.1.3 The Poisson bracket

Let X_H and X_F be Hamiltonian vector fields. Then there exists a smooth function $G \in C^\infty(M)$ such that

$$[X_H, X_F] = X_G.$$

The *Poisson bracket* H, F of smooth functions H and F on M is the derivation of the function F in the direction of the phase flow with Hamiltonian H.

$$\{H, F\}(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(g_H^t(x)).$$

We have that

$$[X_H, X_F] = X_{\{H,F\}}$$

and

$$\{F, H\} = \mathrm{d}F(X_H) = \omega(X_H, X_f).$$

A functions F is a first integral of the phase flow with Hamiltonian H if and only if its Poisson bracket with H is identically zero.

The Poisson bracket is skewsymmetric,

$$\{F, H\} = -\{H, F\},$$

and satisfies the Jacobi identity,

$$\big\{\{F, H\}, G\big\} + \big\{\{G, F\}, H\big\} + \big\{\{H, G\}, F\big\}.$$

Therefore, it makes the vector space of smooth function on a manifold M into a Lie algebra. The map $H \mapsto X_H$ defines a morphism of the Lie algebra of smooth functions on M (with the Poisson bracket) to the Lie algebra of vector fields (with the bracket given by the commutator).

2.2 Liouville's theorem

Let H be a smooth function on M. Then the corresponding Hamiltonian vector field X_H defines a dynamical system, which is called the *Hamiltonian dynamical system with Hamiltonian* H.

Two function F_1 and F_2 on (M,ω) are *in involution* if $\{F_1,F_2\}=0$.

Theorem (Liouville)**.** *Suppose $F_1,\dots,F_n$ are in involution on (M,ω). For $f=(f_1,\dots,f_n)\in\mathbb{R}^n$, consider a level set of the function $(F_1,\dots,F_n)$*

$$M_f=\big\{x\in M \mid \; F_i(x)=f_i\big\}.$$

Assume that the functions F_i are independent on M_f, i.e. the 1-forms $\mathrm{d}F_i$ are linearly independent at each point of M_f. Then

(1) *M_f is a smooth manifold invariant under the phase flow with Hamiltonian $H:=F_1$.*

(2) *If the manifold M_f is compact and connected, then it is diffeomorphic to the n-dimensional torus.*

(3) *The Hamiltonian dynamical system with Hamiltonian H can be integrated by quadratures.*

3 The noncommutative case

In this section we discuss a possible generalization of the Hamiltonian formalism to the noncommutative case.

3.1 Drinfeld–Jimbo quantum groups

In this section we recall basic material about Drinfeld–Jimbo quantised universal enveloping algebras (introduced in [7,12]) and their representation theory. We refer the reader to [6,9,13] for more details.

3.1.1 Drinfeld–Jimbo quantised universal enveloping algebras $U_q(\mathfrak{g})$

Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra of rank r. We fix a Cartan subalgebra $\mathfrak{h}$ with corresponding root system $\Delta\subseteq\mathfrak{h}^*$, where $\mathfrak{h}^*$ denotes the linear dual of $\mathfrak{h}$. Fix a choice of simple roots $\{\alpha_1,\dots,\alpha_r\}$. Denote by $(\,\cdot\,,\,\cdot\,)$ the symmetric bilinear form induced on $\mathfrak{h}^*$ by the Killing form of $\mathfrak{g}$, normalised so that any shortest simple root α_i satisfies $(\alpha_i,\alpha_i)=2$. Let $\{\varpi_1,\dots,\varpi_r\}$ denote the corresponding set of fundamental weights of $\mathfrak{g}$. The Cartan matrix $A=(a_{ij})$ of $\mathfrak{g}$ is the $(r\times r)$-matrix defined by $a_{ij}:=\big(\alpha_i^\vee,\alpha_j\big)$, where $\alpha_i^\vee:=2\alpha_i/(\alpha_i,\alpha_i)$.

Let $q\in\mathbb{C}$ be such that q is not a root of unity and denote $q_i:=q^{(\alpha_i,\alpha_i)/2}$. The *quantised enveloping algebra* $U_q(\mathfrak{g})$ is the noncommutative associative algebra generated by the elements E_i,F_i,K_i, and K_i^{-1} for $i=1,\dots,r$, subject to the relations

$$K_iE_j=q_i^{a_{ij}}E_jK_i,\quad K_iF_j=q_i^{-a_{ij}}F_jK_i,\quad K_iK_j=K_jK_i,$$
$$K_iK_i^{-1}=K_i^{-1}K_i=1,\quad E_iF_j-F_jE_i=\delta_{ij}\,\frac{K_i-K_i^{-1}}{q_i-q_i^{-1}}\,,$$

along with the *quantum Serre relations*

$$\sum_{s=0}^{1-a_{ij}}(-1)^s\begin{bmatrix}1-a_{ij}\\ s\end{bmatrix}_{q_i}E_i^{1-a_{ij}-s}E_jE_i^s=0\;\text{ for }\; i\neq j,$$
$$\sum_{s=0}^{1-a_{ij}}(-1)^s\begin{bmatrix}1-a_{ij}\\ s\end{bmatrix}_{q_i}F_i^{1-a_{ij}-s}F_jF_i^s=0\;\text{ for }\; i\neq j,$$

where we have used the q-binomial coefficients defined as follows

$$[n]_q! := [n]_q[n-1]_q \cdots [2]_q[1]_q, \quad \text{where } [k]_q := \frac{q^k - q^{-k}}{q - q^{-1}},$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

A Hopf algebra structure is defined on $U_q(\mathfrak{g})$ by

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$
$$S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i, \quad S(K_i) = K_i^{-1},$$
$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.$$

Let $\mathcal{P}$ be the weight lattice of $\mathfrak{g}$, and $\mathcal{P}^+$ its set of dominant integral weights. We consider $\mathsf{Rep}_1\, U_q(\mathfrak{g})$, the full sub-category of the category of (left) $U_q(\mathfrak{g})$-modules, whose the objects are finite-dimensional $U_q(\mathfrak{g})$-modules having a weight decomposition $V = \bigoplus_{\mu \in \mathcal{P}} V(\mu)$. Recall that a vector $v \in V$ is called a *weight vector* of *weight* $\mu \in \mathfrak{h}^*$ if $K_i \triangleright v = q^{(\alpha_i, \mu)} v$ for all $i = 1, \ldots, r$. The category $\mathsf{Rep}_1\, U_q(\mathfrak{g})$ is a semisimple tensor category whose simple objects are irreducible modules V_λ with the highest weight $\lambda \in \mathcal{P}^+$. Moreover, the character of V_λ is given by the classical Weyl character formula for the irreducible $\mathfrak{g}$-module $\widehat{V}_\lambda$ with highest weight λ. In fact, the category $\mathsf{Rep}_1\, U_q(\mathfrak{g})$ is equivalent to the category $\mathcal{O}_f$ of finite-dimensional representations of $\mathfrak{g}$. We refer to [8, § 5.8] and [13, § 7] for further details.

3.1.2 Quantum coordinate algebras

In this subsection we recall some necessary material about quantised coordinate algebras, see [13, § 6 and § 7] and [17] for further details. Let V be a finite-dimensional left $U_q(\mathfrak{g})$-module, $v \in V$, and $f \in V^*$, the $\mathbb{C}$-linear dual of V, endowed with its right $U_q(\mathfrak{g})$-module structure. An important point to note is that, with respect to the equivalence of left and right $U_q(\mathfrak{g})$-modules given by the invertible antipode, the left module corresponding to V_μ^* is isomorphic to $V_{-w_0(\mu)}$, where w_0 denotes the longest element in the Weyl group of $\mathfrak{g}$.

Consider the function $c^V_{f,v} : U_q(\mathfrak{g}) \to \mathbb{C}$ defined by

$$c^V_{f,v}(X) := f(X \triangleright v).$$

The *coordinate ring* of V is the subspace

$$C(V) := \operatorname{Span}_{\mathbb{C}} \left\{ c^V_{f,v} \mid v \in V, \ f \in V^* \right\} \subseteq U_q(\mathfrak{g})^\circ,$$

where $U_q(\mathfrak{g})^\circ$ denote the Hopf dual of $U_q(\mathfrak{g})$. A $U_q(\mathfrak{g})$-bimodule structure on $C(V)$ is given by

$$(Y \triangleright c^V_{f,v} \triangleleft Z)(X) := f((ZXY) \triangleright v) = c^V_{f \triangleleft Z, Y \triangleright v}(X).$$

It is easily checked that $C(V) \subseteq U_q(\mathfrak{g})^\circ$, and moreover that a Hopf subalgebra of $U_q(\mathfrak{g})^\circ$ is given by

$$\mathcal{O}_q(G) := \bigoplus_{\mu \in \mathcal{P}^+} C(V_\mu).$$

We call $\mathcal{O}_q(G)$ the *quantum coordinate algebra of* G, where G is the compact, connected, simply-connected, simple Lie group having $\mathfrak{g}$ as its complexified Lie algebra.

3.2 The quantum flag manifolds

Let $\{\alpha_i\}_{i \in S}$ be a subset of simple roots. In what follows, by abuse of notation, we denote by S not only an index subset but also the corresponding subset of simple roots $\{\alpha_i\}_{i \in S}$. Consider the Hopf subalgebra

$$U_q(\mathfrak{l}_S) := \left\langle K_i, E_j, F_j \mid i = 1, \ldots, r; j \in S \right\rangle.$$

By construction $\mathcal{O}_q(G)$ is a $U_q(\mathfrak{g})$-module, hence, it is a $U_q(\mathfrak{l}_S)$-module too.

The *quantum flag manifold associated to* S is the space of $U_q(\mathfrak{l}_S)$-invariants in $\mathcal{O}_q(G)$, and is denoted by

$$\mathcal{O}_q(G/L_S) := \mathcal{O}_q(G)^{U_q(\mathfrak{l}_S)},$$

where L_S is the Levi subgroup of G corresponding to S.

Let S be a subset of simple roots of $\mathfrak{g}$. Following the classical case (see, for example, [5]) we say that the quantum flag manifold associated to S is of *irreducible type* if $\mathfrak{g}/\mathfrak{l}_S$ is a direct sum of two dual irreducible $\mathfrak{l}_S$-modules.

3.2.1 The Heckenberger–Kolb calculus

A *differential calculus* $(\Omega^\bullet \simeq \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \Omega^k, \mathrm{d})$ is a differential graded algebra (dg-algebra) which is generated in degree 0 as a dg-algebra, that is to say, it is generated as an algebra by the elements $a, \mathrm{d}b$ for $a, b \in \Omega^0$. We call an element $\omega \in \Omega^\bullet$ a *form*, and if $\omega \in \Omega^k$ for some $k \in \mathbb{N}$, then ω is said to be *homogeneous* of degree $|\omega| := k$. For a given algebra B, a *differential calculus over* B is a differential calculus such that $\Omega^0 = B$.

Let ${}^A_B\mathsf{Mod}_0$ be the category whose objects are left A-comodules $\Delta_L : \mathcal{F} \to A \otimes \mathcal{F}$, endowed with a B-bimodule structure, such that

1. $\Delta_L(bf) = \Delta_L(b)\Delta_L(f)$ for all $f \in \mathcal{F}$, $b \in B$,
2. $\mathcal{F}B^+ = B^+\mathcal{F}$,

and whose morphisms are left A-comodule, B-bimodule, maps. In the case when $A = \mathcal{O}_q(G)$ and $B = \mathcal{O}_q(G/L_S)$, an object in the category ${}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathsf{Mod}_0$ corresponds to a noncommutative generalization of a module of smooth sections of a homogeneous vector bundle over G/L_S.

As it was shown in [10, Theorem 7.2] and [11, Propositions 3.6 and 3.7] for any irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$, there exist exactly two non-isomorphic, left $\mathcal{O}_q(G)$-covariant, finite-dimensional differential calculi

$$\Omega_q^{(\bullet,0)}(G/L_S),\ \ \Omega_q^{(0,\bullet)}(G/L_S) \in {}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathsf{Mod}_0,$$

of the classical dimension.

In [15], Matassa showed (see also [16]) that for the irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$ there exists a 2-form

$$\omega \in (\Omega_q^\bullet(G/L_S))^{\mathrm{co}(\mathcal{O}_q(G))}$$

such that it is closed

$$\mathrm{d}\omega = 0$$

and nondegenerate

$$\underbrace{\omega \wedge \cdots \wedge \omega}_{\dim(G/L_S)\text{ times}} \neq 0,$$

where $\dim(G/L_S)$ is the classical dimension of G/L_S.

Therefore, we can consider ω as a *noncommutative symplectic structure* on $\mathcal{O}_q(G/L_S)$.

3.2.2 Yetter–Drinfeld modules

A *braiding* on a monoidal category C is a natural isomorphism σ between the functors $-\otimes-$ and $-\otimes^{\mathrm{opp}}-$ such that the hexagonal diagrams commute, see [8, § 8.1] for details. A *braided* monoidal category is a pair consisting of a monoidal category and a braiding.

An important example of a braided monoidal category is the category of (right) *Yetter–Drinfeld modules* V over a Hopf algebra H, which are those right H-modules V, with an action $\triangleleft$, and a right H-comodule structure such that

$$v_{(0)} \triangleleft h_{(1)} \otimes v_{(1)}h_{(2)} = (v \triangleleft h_{(2)})_{(0)} \otimes h_{(1)}(v \triangleleft h_{(2)})_{(1)},\ \ h \in H,\ \ v \in V.$$

We denote the category of Yetter–Drinfeld modules, endowed with its obvious monoidal structure, by YD_H^H. A braiding for the category is defined by

$$\sigma : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto w_{(0)} \otimes v \triangleleft w_{(1)} \ \text{ for } \ v \in V, \ \ w \in W.$$

Note that for any $V \in \mathsf{YD}_H^H$, the tensor algebra $\mathcal{T}(V)$ is a braided Hopf algebra in YD_H^H with

$$\Delta(v) := v \otimes 1 + 1 \otimes v, \quad S(v) := -v, \quad \epsilon(v) := 0 \ \text{ for } \ v \in V.$$

3.3 Nichols algebras

For a detailed introduction on Nichols algebras we refer to the surveys [1] and [2]. Let $\mathbf{B}_n$ denote the *braid group on n strands*, which can be described as the group generated by $n-1$ elements $\beta_1, \dots, \beta_{n-1}$ subject to the relations

$$\begin{aligned} \beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1}, \ \ 1 \le i \le n-2, \\ \beta_i\beta_j = \beta_j\beta_i, \ \ 1 \le i, j \le n-2, \ \ |i-j| \ge 2. \end{aligned}$$

For any $V \in \mathsf{YD}_H^H$ which is finite-dimensional as a vector space, we obtain a representation of the braid group on n strands

$$\rho_n : \mathbf{B}_n \to GL(V^{\otimes n}),$$

given by

$$\rho_n(\beta_i) = \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \sigma \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id},$$

where σ is acting on $V \otimes V$ in position i and $i+1$.

There is a canonical surjective group homomorphism onto the symmetric group $\mathbf{S}_n$,

$$\varphi_n : \mathbf{B}_n \to \mathbf{S}_n,$$

which maps β_i to the simple transposition $\tau_i = (i, i+1)$. Let $\ell(g)$ denote the length of an element $g \in \mathbf{S}_n$. The projection φ_n admits a set-theoretic section, called the *Matsumoto section*

$$s_n : \mathbf{S}_n \to \mathbf{B}_n,$$

which is determined by $s_n(\tau_i) = \beta_i$ and $s_n(\tau_i\tau_{i+1}) = s_n(\tau_i)s_n(\tau_{i+1})$, for $1 \le i \le n$, and $s_n(gf) = s_n(g)s_n(f)$ if $\ell(gf) = \ell(g) + \ell(f)$ for $g, f \in \mathbf{S}_n$. Note that s_n is *not* a group homomorphism.

The *braided symmetrizer* is given by the map

$$\mathfrak{S}_n^\sigma(V) := \sum_{g \in \mathbf{S}_n} \rho_n(s_n(g)) : V^{\otimes n} \to V^{\otimes n}.$$

Moreover, we denote

$$\ker \mathfrak{S}^\sigma(V) := \bigoplus_{n \in \mathbb{Z}_{\ge 0}} \ker \mathfrak{S}_n^\sigma(V).$$

The *Nichols algebra* of V is the braided Hopf algebra in YD_H^H defined by

$$\mathfrak{B}(V) := \mathcal{T}(V) \big/ \ker \mathfrak{S}^\sigma(V).$$

The Nichols algebra $\mathfrak{B}(V)$ has a unique $\mathbb{Z}_{\ge 0}$-grading

$$\mathfrak{B}(V) \simeq \bigoplus_{n \in \mathbb{Z}_{\ge 0}} \mathfrak{B}_n(V), \quad \mathfrak{B}_n(V) := \mathcal{T}^n(V) \big/ \ker \mathfrak{S}_n^\sigma(V),$$

since $\ker \mathfrak{S}^\sigma(V)$ is a homogeneous ideal of $\mathcal{T}(V)$.

Let $B = \mathfrak{B}(V)$ be a Nichols algebra. Since B is a left and right comodule over itself via the comultiplication, it becomes a left and right module over B^*. Following [3], define the *quantum contraction operators* $i^L, i^R : B^* \to \mathrm{End}(B)$ as the representation associated to those actions:

$$i^L_f(b) = f(b_{(1)})b_{(2)}, \quad i^R_f(b) = f(b_{(2)})b_{(1)} \text{ for } b \in B, \ f \in B^*.$$

Consider the $\mathfrak{sl}_n$-series of irreducible quantum flag manifolds, namely, the quantum Grassmannians. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and let m be an integer such that $1 \le m \le n$. Fix $S = \{\alpha_i \mid i \in \{1, \dots, n\} \setminus \{m\}\}$ a subset of simple roots of $\mathfrak{g}$. In this case $\mathfrak{l}_S = \mathfrak{gl}_m \oplus \mathfrak{sl}_{n-m}$. The quantum flag manifold associated to S is called *the quantum* (n,m)*-Grassmannian* and denoted by $\mathcal{O}_q(\mathrm{Gr}_{n,m})$.

As it was shown in [14] for the quantum Grassmannians $\mathcal{O}_q(\mathrm{Gr}_{n,m})$, the Heckenberger–Kolb calculi $\Omega_q^{(\bullet,0)}(\mathrm{Gr}_{n,m})$ and $\Omega_q^{(0,\bullet)}(\mathrm{Gr}_{n,m})$ are Nichols algebras.

3.4 Noncommutative Hamiltonian dynamical systems: an open problem

Here we discuss possible ingredients of a not yet defined noncommutative Hamiltonian dynamical system.

1. As a natural example of manifold in the noncommutative case we propose to consider the quantum Grassmannians $\mathcal{O}_q(\mathrm{Gr}_{n,m})$. Recall, that the algebra $\mathcal{O}_q(\mathrm{Gr}_{n,m})$ admits a C^*-algebraic completion $\mathbb{C}_q(\mathrm{Gr}_{n,m})$.

2. For a noncommutative analogue of vector fields we propose to consider the algebra of derivations of $\mathcal{O}_q(\mathrm{Gr}_{n,m})$ (the algebra of unbounded derivations of $\mathbb{C}_q(\mathrm{Gr}_{n,m})$ in the C^*-algebraic setting).

3. For a symplectic structure for $\mathcal{O}_q(\mathrm{Gr}_{n,m})$, one can consider its Heckenberger–Kolb calculi together with the symplectic structure found by Matassa.

4. One can define a noncommutative analogue of the map I, see § 2.1.2, using the quantum contraction operators in the Nichols algebras $\Omega_q^{(\bullet,0)}(\mathrm{Gr}_{n,m})$ and $\Omega_q^{(0,\bullet)}(\mathrm{Gr}_{n,m})$.

5. It will allow to define a noncommutative Hamiltonian vector field in the same way as in the classical situation. If the corresponding derivation of $\mathbb{C}_q(\mathrm{Gr}_{n,m})$ happens to be unbounded, one can integrate it to a one-parameter family of automorphisms of the C^*-algebra $\mathbb{C}_q(\mathrm{Gr}_{n,m})$. Such a one-parameter family of automorphisms will be a noncommutative version of a flow of a Hamiltonian vector field. Such a flow will define a noncommutative version of a Hamiltonian dynamical system.

6. The fixed points in $\mathbb{C}_q(\mathrm{Gr}_{n,m})$ of such flows can be considered as noncommutative analogues of first integrals.

It is an intriguing open problem to formulate an analogue of Liouville's theorem for such noncommutative dynamical systems.

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Approximations of Kasparov Categories of C^*-Algebras

George Nadareishvili

1 Introduction

For a locally compact Hausdorff space X, one may consider the C^*-algebra $C_0(X)$ of continuous functions from X to complex numbers that vanish at infinity. This construction is functorial and makes the category of locally compact Hausdorff spaces contravariantly equivalent to the category of commutative C^*-algebras (see for example [10]). As a result, one can translate back and forth between topological and algebraic concepts.

$$\begin{array}{rcl} \text{topology, } C_0(X) \text{ proper map,} & & \text{algebra, } C^*\text{-algebra } A \ *\text{-homomorphism,} \\ \text{homeomorphism measure,} & & \text{isomorphism positive functional,} \\ \text{compact open subset,} & \longleftrightarrow & \text{unital ideal,} \\ \text{closed subset connected,} & & \text{quotient projectionless,} \\ \text{2nd countable topological K-theory} & & \text{separable operator K-theory} \end{array}$$

That is, each property of a locally compact Hausdorff space X can be formulated in terms of the function algebra $C_0(X)$ and will then usually make sense for any noncommutative C^*-algebra. This way, C^*-algebra theory can successfully be regarded as a kind of "noncommutative topology".

2 Homological techniques for C^*-algebras

2.1 The universal invariant

For simplicity of presentation we suppress degree shifts and grading for functors.

As topological spaces, we wish to study C^*-algebras using homological invariants. Due to the theorem by Higson [4, Theorem 4.5], there exists a universal category KK receiving a functor from C^*-algebras, such that any additive, split short exact sequence preserving functor H into an additive category A, that also inverts the tensor product with the C^*-algebra of compact operators is automatically homotopy-invariant and factors through KK.

$$\begin{tikzcd} C^*\text{-alg} \arrow[rr, "H"] \arrow[dr] & & \mathrm{A} \\ & \mathrm{KK} \arrow[ur] & \end{tikzcd}$$

In this light, the study of the category KK is of a fundamental importance to noncommutative topology. The category KK has separable C^*-algebras as objects and Kasparov KK-groups $\mathrm{KK}(A, B)$ as morphisms, with the Kasparov product as composition (for details see for example [2]). In addition, KK carries the structure of a triangulated category (see [5]).

Generalizing Atiyah–Segal's classical vector bundle K-cohomology of spaces, the identification

$$\mathrm{Hom}(\mathbb{C}, A) = \mathrm{KK}(\mathbb{C}, A) \cong \mathrm{K}(A)$$

presents KK-theory as a natural generalization of K-theory. So, naturally, one wishes to compute KK-theory (and consequentially any sensible homology theory) only using K-theoretic invariants. However, in general, this is hard or impossible to accomplish.

One way to approach this problem is with relative homological algebra. We fix a triangulated category T and a homological functor $I : \mathrm{T} \to \mathrm{A}$ beforehand (in our case this might be K-theory) and consider all homological functors F such that $\ker I \subseteq \ker F$ on morphisms. We call these I-exact homological functors. The idea here is that F is only at most as refined as I.

As it turns out, there is a universal functor among such F.

Definition. I-exact homological functor U is *universal*, if any other I-exact homological functor $G : \mathrm{T} \to \mathrm{A}'$ factors as $G = \bar{G} \circ U$ for an exact functor $\bar{G} : \mathrm{A} \to \mathrm{A}'$ that is unique up to natural isomorphism.

Theorem 2.1 (Beligiannis [1, Section 3]). *For every homological functor I on a triangulated category T, there exists an abelian category $\mathrm{A}_I(\mathrm{T})$ and a universal I-exact homological functor $U : \mathrm{T} \to \mathrm{A}_I(\mathrm{T})$.*

An object $P \in \mathrm{T}$ is called I-projective if $\mathrm{Hom}_{\mathrm{T}}(P, -)$ is an I-exact functor. In favorable cases, when enough I-projective objects are available, we can build a consequent I-relative homological algebra by constructing corresponding I-projective resolutions. Having a universal I-exact stable homological functor U means that the abelian homological algebra in A and I-relative homological algebra in T are the same. More precisely, we have

Theorem 2.2 (Beligiannis [1, Proposition 4.19]). *Let I be a homological functor on a triangulated category T and let $U : \mathrm{T} \to \mathrm{A}$ be a universal I-exact homological functor into an abelian category A. Suppose that idempotent morphisms in T split and that there are enough I-projective objects in T. Then there are enough projective objects in A and U induces an equivalence between the full subcategories of I-projective objects in T and of projective objects in A.*

It is also possible to define derived functors relative to I. Then one can write down a spectral sequence that relates a homological functor to its derived functors. We are only going to recall the case of a Universal Coefficient Theorem, where this spectral sequence collapses to a short exact sequence and we are able to compute the derived functors using the universal I-exact functor.

Theorem 2.3 (Meyer-Nest [6, Theorem 4.4]). *Let I be a homological functor in a triangulated category T and let $U : \mathrm{T} \to \mathrm{A}$ be a universal I-exact homological functor into a abelian category A with enough projective objects. For $A \in \mathrm{T}$, let $U(A)$ have a projective resolution of length 1. Suppose also that A belongs to a smallest triangulated subcategory generated by I-projective objects. Then for any $B \in \mathrm{T}$ there is a natural short exact sequence*

$$\mathrm{Ext}^1_{\mathrm{A}}\big(U(\Sigma A), U(B)\big) \rightarrowtail \mathrm{T}(A, B) \twoheadrightarrow \mathrm{Hom}_{\mathrm{A}}\big(U(A), U(B)\big)\ ,$$

where $\mathrm{Ext}^1_{\mathrm{A}}$ and $\mathrm{Hom}_{\mathrm{A}}$ denote extension and morphism groups in A and Σ is a suspension on T.

When triangulated category in question is KK, K-theory itself turns out to be a universal K-exact homological functor. The suspension Σ is tensoring with $C_0(\mathbb{R})$ and the triangulated subcategory generated by K-projective objets is what is called a *bootstrap class*. Consequently, we get the celebrated Universal Coefficient Theorem by Rosenberg and Schochet

Theorem 2.4 (Rosenberg–Schochet [9]). *Let A be a separable C^*-algebra. Then A is in a bootstrap class if and only if, for all $B \in \mathrm{KK}$, there is a short exact sequence of abelian groups*

$$\mathrm{Ext}^1_{\mathrm{Ab}}\big(\mathrm{K}(\Sigma A), \mathrm{K}(B)\big) \rightarrowtail \mathrm{KK}(A, B) \twoheadrightarrow \mathrm{Hom}_{\mathrm{Ab}}\big(\mathrm{K}(A), \mathrm{K}(B)\big)\ .$$

2.2 Examples of use

2.2.1 Localizing subcategories in the bootstrap class

The relevant notion of a subcategory of a triangulated category T with coproducts is that of a localizing subcategory.

Localizing subcategories form a lattice under intersection and triangulated closure of the union. The question arises: can we classify the lattice of all localizing subcategories Loc for KK? The answer is yes, if we restrict our attention to a smaller bootstrap class $\mathcal{B} \subseteq \mathrm{KK}$ and use homological techniques explained in a previous section. More precisely, using the Universal Coefficient Theorem one gets

Theorem 2.5 (Dell'Ambrogio [3])**.** *There is an inclusion-preserving bijection between localizing subcategories of the Bootstrap class on the one hand, and subsets of the Zariski spectrum* $\operatorname{Spec}(\mathbb{Z})$ *on the other*

$$\operatorname{Loc}(\mathcal{B}) \cong \mathcal{P}\big(\operatorname{Spec}(\mathbb{Z})\big) = \mathcal{P}\big(\{\text{prime numbers and zero}\}\big).$$

2.2.2 Localizing subcategories in the filtered bootstrap class

Similar to KK, one can define a universal triangulated category $\mathrm{KK}(n)$ for filtrations of noncommutative spaces

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n,$$

where A_i is an ideal in A_{i+1}. There is a corresponding bootstrap class $\mathcal{B}(n)$ with the version of the Universal Coefficient Theorem (see [7]).

We wish to classify all localizing subcategories in $\mathcal{B}(n)$. Denote by NC_{n+1} the lattice of noncrossing partitions of the regular $n+1$-gon (see Figure 2.1). Again, using techniques of relative homological algebra, we get

Theorem 2.6 (Nadareishvili [8])**.** *The lattice of localizing subcategories of* $\mathcal{B}(n)$ *is isomorphic to the product of lattices of noncrossing partitions of* $n+1$*-gon over* $\operatorname{Spec}\mathbb{Z}$

$$\operatorname{Loc}\big(\mathcal{B}(n)\big) \cong \prod_{p\in\operatorname{Spec}\mathbb{Z}} \mathrm{NC}^p_{n+1}.$$

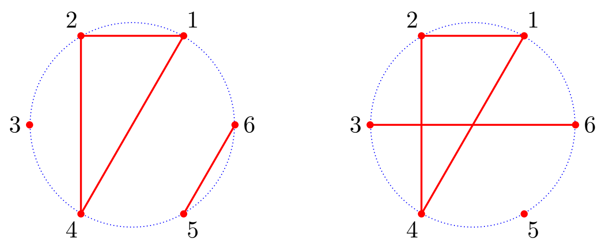

Figure 2.1. The left picture shows the noncrossing partition $\{\{1,2,4\},\{3\},\{5,6\}\}$ of the regular hexagon represented as vertices on a circle. The partition $\{\{1,2,4\},\{3,6\},\{5\}\}$ on the right picture is crossing.

For example, the lattice of localizing subcategories of $\mathcal{B}(3)$ for a fixed p is given in Figure 2.2.

2.2.3 Finite group actions

For an at most countable full subcategory $\mathcal{C} \subseteq \mathrm{T}$ with countable coproducts, let $I_\mathcal{C}$ be the homological functor

$$I_\mathcal{C} : \mathrm{T} \longrightarrow \prod_{C\in\mathcal{C}} \mathrm{Ab}^\mathbb{Z}, \quad A \longmapsto \big(\operatorname{Hom}_\mathrm{T}(C,A)\big)_{C\in\mathcal{C}},$$

where we assume that $I_\mathfrak{C}(A)$ is countable for all $A \in \mathrm{T}$. The enrichment of $I_\mathcal{C}$ to the functor

$$I'_\mathcal{C} : \mathrm{T} \longrightarrow \operatorname{Funct}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})_c,$$

into the abelian category of countable functors, is the universal $I_\mathcal{C}$-exact stable homological functor [7, Theorem 4.4]. In general, the arising spectral sequence that relates a homological functor to its derived functors does not degenerate to a short exact sequence to give a Universal Coefficient Theorem.

For example, we can look at a category KK^G, Kasparov's category of equivariant C^*-algebras with actions of a finite group G. Here, we should take $\mathcal{C}$ to be the set of induced algebras of matrix algebras over subgroups of G. This gives all Type-I C^*-algebras. Even though there is no Universal Coefficient Theorem, the computation of the universal invariant in question can again be done using homological techniques.

One can push this even further and explore the special circumstances under which the Universal Coefficient Theorem is available. To what extent this is possible is a work in progress.

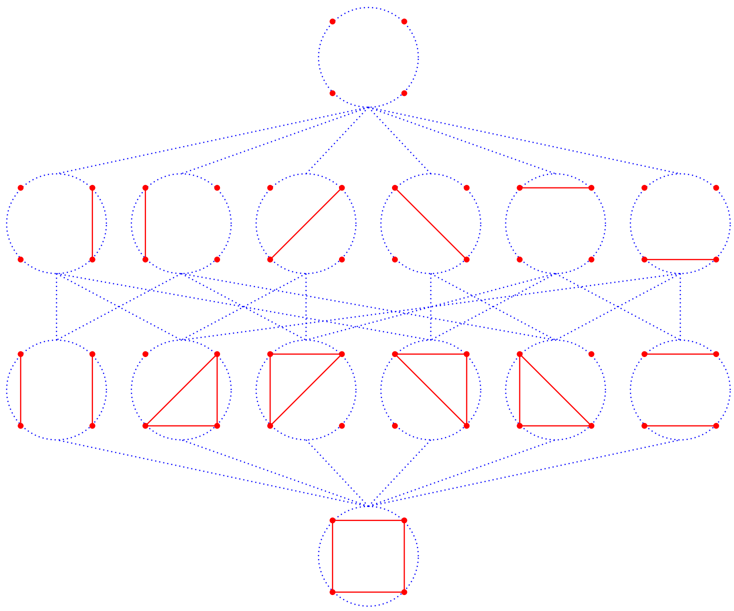

Figure 2.2. Localizing subcategories of $\mathcal{B}(3)$ for a fixed p.

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Smale Spaces and Their C^*-Algebras

Karen R. Strung

Abstract

A Smale spaces is a type of hyperbolic dynamical system introduced by Ruelle to model the behaviour of the restriction of a so-called "Axiom A diffeormorphism" to its basic sets. Examples include shifts of finite type, certain tiling spaces, and Anosov diffeormorphisms. From a Smale space we can construct several C*-algebras which capture different aspects of the dynamics. This note explores the interactions between Smale spaces and C*-algebras. Topological dynamics and Smale spaces are introduced along with several examples. It is shown how the dynamics of a Smale space leads to étale groupoids, which are then used to construct C*-algebras. Properties of the C*-algebras and their relation to properties of the Smale space are investigated. Appendices on the basics of C*-algebras are included.

1 Introduction

The study of C*-algebras has its origins in quantum physics, where they were used to study the unusual noncommutative behaviour of observables at the quantum level. Mathematicians quickly realized that C*-algebras were useful models to encode and study the behaviour of many physical and mathematical objects including symmetries, networks, and systems which evolve over time. Stemming from the pioneering work of Murray and von Neumann on what they called rings of operators, the study of C*-algebras continues to be an active area of research.

Interpreting a mathematical or physical system in C*-algebraic theory typically involves input information from the system in question and an output C*-algebra. While the theory is flexible enough to allow for a wide range of input information, the output C*-algebra has sufficient structure to allow for tractable analysis. In particular, topological dynamical systems – topological spaces equipped with continuous transformations – have long been a source for constructing interesting examples of C*-algebras. The process of encoding the relevant dynamics in a C*-algebra first involves constructing a topological groupoid which allows one to study its groupoid C*-algebra or C*-algebras. The underlying dynamical system influences the structural properties of the C*-algebra while at the same time one hopes to gain information about dynamical systems otherwise inaccessible without the C*-algebraist's toolkit.

Smale spaces are a class of topological dynamical systems that were defined by Ruelle [24] as a way of axiomatizing the behaviour of the basic sets associated to Smale's Axiom A diffeomorphisms [28]. A Smale space is a pair (X, φ), where X is a compact metric space and $\varphi : X \to X$ a homeomorphism, which has a local hyperbolic structure: at every point $x \in X$ there is a small neighbourhood which decomposes into the product of a stable and unstable set. For a Smale space, the dynamical behaviour which we seek to study is the asymptotic behaviour of points. The appropriate C*-algebras, defined by Putnam [18] following earlier work by Ruelle [23] encode this behaviour via groupoid C*-algebras associated to stable, unstable, and homoclinic equivalence relations, respectively.

This note is based on a course given at the summer school *"Algebra, topology and Analysis:* C* *and A_∞ algebras"* which took place in Gonio, Batumi, Georgia in September 2021. It is intended as an introduction to Smale spaces and their C*-algebras. In Section 2 some of the basics of topological dynamics are introduced, in particular recurrence properties and topological entropy. Section 3 focuses specifically on Smale spaces. Before constructing C*-algebras, some material on étale groupoids is developed in Section 4. The final section looks at properties of the resulting groupoid C*-algebras. As this was originally an introductory series of lectures, no previous knowledge of C*-algebras was assumed. Therefore two appendices have been included: The first appendix develops the basics of C*-algebras. The second includes some details about more advanced notions.

2 Dynamics

Although this note concerns itself with Smale spaces, we begin in the more general setting of topological dynamics. Topological dynamical systems are topological spaces equipped with continuous transformations. Here, this means a self-homeomorphism. One can think of a topological dynamical as discrete time evolution of a system. Each application of the homeomorphism correspondence to a step (forward or backward) in time after which one inspects the state of the system. For this reason, we would like to understand what happens to points in the spaces under repeated application of the homeomorphism. With that in mind, we are interested in *recurrence* properties, for example, is a given point fixed by the system? Does it return to a neighbourhood of itself? If so, how often? We are also interested in the *entropy* of the system, which measures its chaos. In this section, we define these concepts. We will see how they relate to Smale spaces in Section 3.

Definition 2.1. A *topological dynamical system* is a pair (X, φ) where X is a compact Hausdorff space and $\varphi : X \to X$ is a homeomorphism.

Given a point $x \in X$, we denote its *orbit* by

$$\mathrm{orb}_\varphi(x) := \{\varphi^n(x) \mid n \in \mathbb{Z}\}.$$

We can also consider the forward orbit of x, $\{\varphi^n(x) \mid n \in \mathbb{N}\}$ and the backward orbit of x, $\{\varphi^{-n}(x) \mid n \in \mathbb{N}\}$.

A point $x \in X$ is called a *fixed point* if $\varphi(x) = x$. It is *periodic* if there exists $n > 0$ such that $\varphi^n(x) = x$. If x is periodic, its *period* is

$$\min\{n \in \mathbb{N} \setminus \{0\} \mid \varphi^n(x) = x\}.$$

Given $n \in \mathbb{N} \setminus \{0\}$, let

$$\mathrm{Per}_n(X, \varphi) := \{x \in X \mid x \text{ has period } n\},$$

the set of periodic points with period n, and let

$$\mathrm{Per}(X, \varphi) := \bigcup_{n>0} \mathrm{Per}_n(X, \varphi),$$

the periodic points of (X, φ).

Examples 2.2.

(1) Let $\varphi : S^2 \to S^2$ be a homeomorphism. It follows from the Hairy Ball theorem that φ must fix the poles or swap them. Thus any such systems has periodic points.

(2) Let $0 \le \theta < 1$ and define $R_\theta : \mathbb{T} \to \mathbb{T}$ by $R_\theta(\lambda) = e^{2\pi i\theta}\lambda$. If $\theta \in \mathbb{Q}$, then every point is periodic. If $\theta \in [0, 1) \setminus \mathbb{Q}$, then there are no periodic points.

Definition 2.3. Let (X, φ) be a topological dynamical system. A point $x \in X$ is called *non-wandering* if, for every open set $U \subset X$ containing x, there exists $n \in \mathbb{N}\setminus\{0\}$ such that $\varphi^n(U) \cap U \neq \varnothing$. Otherwise x is called *wandering*.

Denote by $NW(X, \varphi)$ the set of non-wandering points for the topological dynamical system (X, φ). Note that $x \in NW(X, \varphi)$ if and only if for every open set $U \subset X$ containing x, there exists $z \in U$ and $n \in \mathbb{N} \setminus \{0\}$ such that $\varphi^n(z) \in U$. (If $\varphi^n(U) \cap U$ is non-empty, choose $y \in \varphi^n(U) \cap U$ and let $z = \varphi^{-n}(y)$.)

Clearly every periodic point is non-wandering. If the periodic points are dense in X, then $NW(X, \varphi) = X$.

Proposition 2.4. *Let (X, φ) be a topological dynamical system. Then*

(i) *$NW(X, \varphi)$ is φ-invariant, that is, $\varphi(NW(X, \varphi)) \subset NW(X, \varphi)$,*

(ii) *$NW(X,\varphi)$ is closed,*

(iii) *$NW(X,\varphi)$ is non-empty.*

(Only (iii) *requires that X is compact.)*

Proof. For (i), let $x \in NW(X,\varphi)$. Let V be an open neighbourhood of $\varphi(x)$. Then $\varphi^{-1}(V) = U$ is an open set containing x, so there is $n \in \mathbb{N} \setminus \{0\}$ such that $\varphi(U) \cap U \neq \varnothing$. It follows that

$$\varphi(V) \cap V = \varphi^{n-1}(\varphi(U)) \cap \varphi(U) \neq \varnothing.$$

So $\varphi(x) \subset NW(X,\varphi)$.

For (ii), suppose that $(x_n)_{n\in\mathbb{N}} \subset NW(X,\varphi)$ is a sequence converging to some $x \in X$. Then for any open set U containing x there exists N such that $x_n \in U$ for every $n > N$. Since $x_n \in U$, there exists some $m \in \mathbb{N} \setminus \{0\}$ such that $\varphi^n(U) \cap U \neq \varnothing$. Thus $x \in NW(X,\varphi)$.

Now we prove (iii). Suppose that $NW(X,\varphi)$ is empty. Then if $x_1 \in X$, for every $U_1 \subset X$ containing x_1, we have that $\varphi^n(U_1) \cap U_1 = \varnothing$, for every $n \in \mathbb{N}$. If $X \setminus \bigcup_{n\in\mathbb{N}} \alpha^n(U_1)$ is non-empty then we must have $\alpha^n(U) \cap \alpha^m(U) = \varnothing$ for every $n \neq m \in \mathbb{N}$, since otherwise we could find a non-wandering point in $\alpha^{\min\{m,n\}}(U)$. Since X is compact, we must have $\bigcup_{n\in\mathbb{N}} \alpha^n(U_1) = X$. Thus finitely many $\alpha^n(U)$ cover X, so there must be $m \in \mathbb{N}$ such that $\alpha^n(U) \cap \alpha^m(U) \neq \varnothing$, a contradiction. So $NW(X,\varphi)$ is non-empty. □

If $NW(X,\varphi) = X$, then we say that (X,φ) is non-wandering.

Definition 2.5. Let (X,φ) be a topological dynamical system.

(1) (X,φ) is *irreducible* if, for every pair U, V of open sets there exists $N \in \mathbb{N} \setminus \{0\}$ such that $\varphi^N(U) \cap V \neq \varnothing$.

(2) (X,φ) is *irreducible* if, for every pair U, V of open sets there exists $N \in \mathbb{N} \setminus \{0\}$ such that $\varphi^n(U) \cap V \neq \varnothing$ for every $n \geq N$.

It is clear that if (X,φ) is mixing, then it is also irreducible, and if it is irreducible then it is also non-wandering. However, none of the reverse implications hold. A very simple example is the following. Let $X = \{0,1\}$ and define $\varphi : X \to X$ by $\varphi(0) = 1$ and $\varphi(1) = 0$. Then (X,φ) is non-wandering and irreducible, but is not mixing.

There are various notions of what it means for two topological dynamical systems to be "the same". The strongest of these is *conjugacy.*

Definition 2.6. Topological dynamical systems (X,φ) and (Y,ψ) are *conjugate* if there exists a homeomorphism $h : X \to Y$ such that $h \circ \varphi = \psi$ and hence also $h^{-1} \circ \psi = \varphi$. The homeomorphism h is a called a *conjugacy.*

An important conjugacy invariant for topological dynamical systems is its *topological entropy.* Loosely speaking, this measures how choatic the dynamical system is by examining the complexity of the orbit structure.

Let (X,d) be a compact metric space and $\varphi : X \to X$ a homeomorphism. Let $N \in \mathbb{Z}_{\geq}0$ and define

$$d_N(x,y) = \max_{0\leq j<N} d(\varphi^j(x), \varphi^j(y)).$$

If $\mathcal{U}$ is a collection of subsets of X then we call $\mathcal{U}$ an (N,ϵ)-*cover* if $\mathcal{U}$ is a cover of X and for each $A \in \mathcal{U}$, we have $d_N(x,y) < \epsilon$ for every $x, y \in A$. In other words, the diameter of A with respect to d_N is less than ϵ. Let $|\mathcal{U}|$ denote the cardinality of a cover $\mathcal{U}$, and set

$$\operatorname{cov}(N,\epsilon,\varphi) := \min\big\{|\mathcal{U}| \mid \mathcal{U} \text{ is an } (N,\epsilon)\text{-cover of } X\big\}.$$

Definition 2.7. Let (X, φ) be a topological dynamical system where X is a compact metric space. Then the *topological entropy* of (X, φ) is

$$h(\varphi) := \lim_{\epsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log(\mathrm{cov}\,(n, \epsilon, \varphi)) \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

A set $A \subset X$ is called (N, ϵ)-*spanning* if, for every $x \in X$, there exists $y \in A$ such that $d_N(x, y) < \epsilon$. Since X is compact, there always exists (N, ϵ)-spanning sets which are finite. Let $|A|$ denote the cardinality of the set A. Define

$$\mathrm{span}\,(N, \epsilon, \varphi) := \min\{|A| \mid A \text{ is } (N, \epsilon)\text{-spanning}\}.$$

A $A \subset X$ is (N, ϵ)-*separated* if, for every $x, y \in A$ we have $D_N(x, y) > \epsilon$. Define

$$\mathrm{sep}\,(N, \epsilon, \varphi) := \big\{|A| \mid A \text{ is } (N, \epsilon)\text{-separated}\big\}.$$

These allow for the following equivalent definitions of topological entropy [3].

Proposition 2.8. *Let (X, φ) be a topological dynamical system where X is a compact metric space. Then*

$$h(\varphi) = \lim_{\epsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log(\mathrm{span}\,(n, \epsilon, \varphi)) = \lim_{\epsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log(\mathrm{sep}\,(n, \epsilon, \varphi)).$$

3 Smale spaces

A Smale space (X, φ) is a dynamical system with a particulary nice local hyperbolic structure. Before giving the (rather unintuitive) definition, let us consider some examples of Smale spaces.

Example 3.1 (Edge shift of a finite directed graph)**.** Let $G = (V, E)$ be a finite directed graph, that is, V is a finite set of *vertices* and E a finite set of *edges*, and there are maps $r, s : E \to V$, respectively called the *range* and *source* maps, which make the edges directed. We think of an edge e with $s(e) = v_1$ and $r(e) = v_2$ as an arrow from v_1 to v_2. For example, the graph in Figure 3.1 is given by $V = \{v_0, v_1, v_2\}$ and $E = \{e_0, e_1, e_2, e_3, e_4\}$, with $v_0 = s(e_0) = r(e_0) = s(e_1) = s(e_2) = r(e_3)$, $v_1 = r(e_1) = s(e_4)$, and $v_2 = r(e_2) = s(e_3)$.

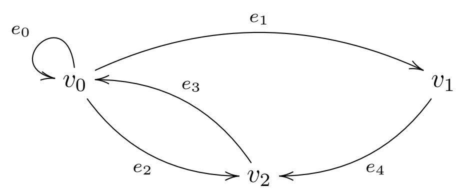

Figure 3.1. A directed graph.

The *bi-infinite path space* of G is

$$X_G := \Big\{(x_n)_{n \in \mathbb{Z}} \subset E^{\mathbb{Z}} \mid s(x_{n+1}) = r(x_n) \text{ for all } n \in \mathbb{Z}\Big\},$$

together with the topology induced by the metric

$$d(x, y) = \inf\big\{2^{-n} \mid x_j = y_j \text{ for every } n < j < n\big\},$$

where $x = (x_n)_{n \in \mathbb{Z}} \in X_G$ and $y = (y_n)_{n \in \mathbb{Z}} \in X_G$. The metric space (X_G, d) is a Cantor space. We equip X_G with the *left shift* map,

$$\sigma : X_G \to X_G, \quad x \mapsto \sigma(x),$$

where

$$\sigma(x)_n = x_{n+1}, \;\; n \in \mathbb{Z}.$$

Let $\epsilon = k^{-1}$ for $k \geq 1$ and let $x = (x_n)_{n\in\mathbb{Z}} \in X_G$. Let $B(x,\epsilon)$ denote the open ball of radius ϵ centred at x. Define

$$X_G^s(x,\epsilon) := \{y \mid \ y_j = x_j \text{ for every } j \geq -k\}.$$

Let $y, z \in X_G^s(x,\epsilon)$. Then $x_j = y_j = z_j$ for every $-k \leq j \leq k$, so $y, z \in B(x,\epsilon)$, and $d(y,z) \leq \epsilon$. If $y \neq z$, then there must be some m such that $x_{-m-1} \neq y_{-m-1}$ but $x_j = y_j$ for every $j \geq m$. So $d(x,y) = 2^{m+1}$. Now let us apply the shift map. We have

$$\sigma(y)_n = y_{n+1} = z_{n+1} = \sigma(z)_n,$$

whenever $n+1 > -m-1$. Thus

$$d(\sigma(y),\sigma(z)) = \frac{1}{2}\, d(y,z).$$

Thus

$$d(\sigma^n(y),\sigma^n(z)) \to 0 \ \text{ as } \ n \to \infty.$$

Now define

$$X_G^u(x,\epsilon) := \{y \mid \ y_j = x_j \text{ for every } j \leq -k\}.$$

Again we have $y, z \in B(x,\epsilon)$ if $y, z \in X_G^u(x,\epsilon)$. However, a similar argument to the above shows that, for $y \neq z$, we see the opposite of what we observed for points in $X_G^s(x,\epsilon)$. In particular,

$$d(y,z) \leq \frac{1}{2}\, d(\sigma(y),\sigma(z),$$

or, since σ is invertible,

$$d(\sigma^{-1}(y),\sigma^{-1}(z)) < \frac{1}{2}\, d(y,z).$$

Thus

$$d(\sigma^{-n}(y),\sigma^{-n}(y)) \to 0 \ \text{ as } \ n \to \infty.$$

Observe that $y \in X^s(x,\epsilon) \cap X^u(x,\epsilon)$ if and only if $y = x$.

Now suppose that $y \in B(x,\epsilon)$. Define $y^s, y^u \in X_G$ by

$$y_n^s := \begin{cases} x_n, & j \geq -k \\ y_n, & j < -k, \end{cases} \qquad y_n^u := \begin{cases} x_n, & j \leq k \\ y_n, & j > k. \end{cases}$$

Then $y^s \in X_G^s(x,\epsilon)$ and $y^u \in X_G^u(y,\epsilon)$.

It follows that $B(x,\epsilon)$ can be given a local coordinate system where our axes are $X^s(x,\epsilon)$ and $X^u(x,\epsilon)$ with origin at the point x. Along the $X^s(x,\epsilon)$, the homeomorphism σ moves points towards each other, while along the $X^u(x,\epsilon)$, σ moves points further apart.

Of course a point y need not be in $B(x,\epsilon)$ for the limit $\lim\limits_{n\to\infty} d(\sigma^n(x),\sigma^n(y))$ to be zero. For this to occur, it is enough that there is some N such that $x_n = y_n$ for every $n \geq N$. We set

$$X_G^s(x) := \{y \in X \mid \ d(\sigma^n(x),\sigma^n(y)) \to 0 \text{ as } n \to \infty\},$$

and

$$X_G^u(x) := \{y \in X \mid \ d(\sigma^{-n}(x),\sigma^{-n}(y)) \to 0 \text{ as } n \to \infty\}.$$

Unlike for the local sets, however, we certainly don't have that $X_G \cong X^s(x) \times X^u(x)$.

An edge shift is an example of a *shift of finite type*, which is defined as follows. Let $\mathcal{A}$ be a finite alphabet of symbols. By a *word* in $\mathcal{A}$ we mean a finite sequence of elements of $\mathcal{A}$. If $w = a_1a_2\ldots a_n$ is a word in $\mathcal{A}$, we say that a sequence $x \subset \mathcal{A}^{\mathbb{Z}}$ contains w if there exists $j \in \mathbb{Z}$ such that $x_{j+i} = a_i$, $1 \leq i \leq n$.

Let $\mathcal{F}$ be a finite set of words in $\mathcal{A}$, which we call *forbidden* words, the reason for which will become clear momentarily. Define

$$X_{\mathcal{F}} := \{x \in X_{\mathcal{F}} \mid \ x \text{ contains no words in } \mathcal{F}\}.$$

Thus any sequence containing a forbidden word is excluded from the set $X_{\mathcal{F}}$. The metric and shift maps are defined the same as for the edge shift above. The system $(X_{\mathcal{F}}, \sigma)$ is called a shift of finite type.

In the case of an edge shift on a graph $G = (V, E)$, our alphabet is given by the edge set E and the set of forbidden words consists of words $e_i e_j$ where $r(e_i) \neq s(e_j)$. Thus the set of forbidden words for the edge shift corresponding to the graph in Figure 3.1 is

$$\mathcal{F} := \big\{e_0e_3, e_0e_4, e_1e_0, e_1e_2, e_1e_3, e_2e_0, e_2e_1, e_2e_4, e_3e_4, e_4e_0, e_4e_1, e_4e_2\big\}.$$

Example 3.2 (Hyperbolic toral automorphisms)**.** Let A be a 2×2 matrix with integer entries, $|\det(A)| = 1$, and eigenvalues not equal to ± 1. Since $|\det(A)| = 1$, we have that A is invertible and $A\mathbb{Z}^2 \subset \mathbb{Z}^2$, so there is a well-defined homeomorphism of the 2-torus,

$$\varphi : \mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{T}^2, \;\; v \mapsto Av \mod \mathbb{Z}^2.$$

For example, the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

satisfies the conditions above. The corresponding homeomorphism of $\mathbb{T}^2$ is the well-known *Arnold's cat map*.

Any such matrix A has eignevalues of the form

$$|\lambda_s| = \mu, \quad |\lambda_u| = \frac{1}{\mu},$$

for some $0 < \mu < 1$. Let v_s be a normalized eigenvector corresponding to the eigenvalue λ_s, and v_u a normalized eigenvector corresponding to λ_u.

Let $v \in \mathbb{T}^2$ and $\epsilon < 1/2$. Define

$$(\mathbb{T}^2)^s(v, \epsilon) := \Big\{w \in \mathbb{T}^2 \mid \; w = v + tv_s \mod \mathbb{Z}^2 \text{ for some } |t| < \epsilon\Big\},$$

and

$$(\mathbb{T}^2)^u(v, \epsilon) := \Big\{w \in \mathbb{T}^2 \mid \; w = v + tv_u \mod \mathbb{Z}^2 \text{ for some } |t| < \epsilon\Big\}.$$

If $w, z \in (\mathbb{T}^2)^s(v, \epsilon)$ where $w = v + t_1 v_s \mod \mathbb{Z}^2$, $z = v + t_2 v_s \mod \mathbb{Z}^2$ then $d(w, z) = |t_1 - t_2|$ while $d(\varphi(w)\varphi(z)) = \lambda^s|t_1 - t_2|$. Thus $d(\varphi(v), \varphi(w)) \leq \mu d(w, z)$. On the other hand, if

$$w = v + t_1 v_u \mod \mathbb{Z}^2, \; z = v + v + t_2 v_u \mod \mathbb{Z}^2 \in (\mathbb{T}^2)^u(v, \epsilon),$$

then

$$d(\varphi^{-1}(w), \varphi^{-1}(z)) \leq \mu d(w, z).$$

A hyperbolic toral automorphisms is an example of an *Anosov diffeomorphism*, which is in turn an example of an *Axiom A diffeormorphism*.

Definition 3.3. A *Smale space* consists of a compact metric space (X, d) together with a homeomorphism $\varphi : X \to X$, constants $\epsilon_X > 0$, $0 < \lambda < 1$, and a *bracket map*

$$[\,\cdot\,,\,\cdot\,] : \Delta_{\epsilon_X} := \big\{(x, y) \in X \times X \mid \; d(x, y) \leq \epsilon_X\big\} \to X$$

which is continuous and satisfies

B1. $[x, x] = x$ for every $x \in X$

B2. $[x, [y, z]] = [x, z]$ for every $x, y, z \in X$ whenever defined,

B3. $[[x, y], z] = [x, z]$ for every $x, y, z \in X$ whenever defined,

B4. $[\varphi(x), \varphi(y)] = \varphi([x, y])$ for every $x, y \in X$ whenever defined, and for $y, z \in X$ such that $x \neq y$, and $d(x, y), d(x, z) \leq \epsilon_X$,

C1. if $[y, x] = x = [z, x]$, then $d(\varphi(y), \varphi(z)) \leq \lambda d(y, z)$,

C2. if $[x, y] = [x, z]$, then $d(\varphi^{-1}(y), \varphi^{-1}(z)) \leq \lambda d(y, z)$.

For example, suppose that $X_{\mathcal{F}}$ is the full 2-shift. This is a shift of finite type with alphabet $\mathcal{A} = \{0, 1\}$ and no forbidden words. Equivalently, it is the edge shift on the directed graph $G_2 = (V = \{v\}, E = \{e_0, e_1\})$, as shown in Figure 3.2.

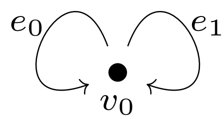

Figure 3.2. Graph of the full 2-shift.

Thus $X = \{0, 1\}^{\mathbb{Z}}$ and $\sigma : X \to X$ is the left shift. Let $\epsilon_X := 1/2$ and $\lambda_X = 1/2$. We define the bracket map

$$[\,\cdot\,,\,\cdot\,] : \Delta_{1/2} \to X,$$

as follows: For $x = (x_n)_{n\in\mathbb{Z}}$ and $y = (y_n)_{n\in\mathbb{Z}}$, set

$$[x, y]_n := \begin{cases} x_n, & n \leq 0, \\ y_n, & n \geq 0. \end{cases}$$

Definition 3.4. Let $((X, d), \epsilon_X, \lambda_X, [\,\cdot\,,\,\cdot\,])$ be a Smale space. For $x \in X$ and $0 < \epsilon \leq \epsilon_X$ define the *local stable set* around x of size ϵ to be

$$X^s(x, \epsilon) := \big\{y \in X \mid \ d(x, y) < \epsilon,\ [y, x] = x\big\},$$

and the *local unstable set* around x of size ϵ to be

$$X^u(x, \epsilon) = \big\{y \in X \mid \ d(x, y) < \epsilon,\ [x, y] = x\big\}.$$

Remark 3.5. Suppose that $((X, d), \varphi, \epsilon_X, \lambda_X, [\,\cdot\,,\,\cdot\,])$ is a Smale space. Consider the 5-tuple $((X, d), \varphi^{-1}, \epsilon_X, \lambda, \{\,\cdot\,,\,\cdot\,\})$, where $\{x, y\} = [y, x]$ for every $x, y \in X$. It is easy to check that this is also a Smale space and that

$$X^s_{\varphi^{-1}}(x, \epsilon) = X^u_{\varphi}(x, \epsilon) \ \text{ and } \ X^u_{\varphi^{-1}}(x, \epsilon) = X^u_{\varphi}(x, \epsilon),$$

for every $0 < \epsilon \leq \epsilon_X$ and every $x \in X$. This relatively easy observation is quite useful: it means that there is usually no loss of generality in considering only the local stable (or local unstable) sets of a Smale space. It is also straightforward to check that, for any $n \in \mathbb{Z}_{>0}$, the dynamical system (X, φ^n) is a Smale space. We leave it as an exercise to determine what the bracket map and Smale space constants are in this case.

Now we will prove some simple facts about the behaviour of the bracket map.

Lemma 3.6. *Let (X, φ) be a Smale space. Suppose that $x, y \in X$ satisfy $d(x, y) < \epsilon_X$. Then*

(1) *$[x, y] = x$ if and only if $[y, x] = y$,*

(2) *$[x, y] = y$ if and only if $[y, x] = x$.*

Proof. If $[x, y] = x$ then $[y, x] = [y, [x, y]]$. By B2, $[y, [x, y]] = [y, y]$ and by B1, $[y, y] = y$. If $[y, x] = y$ then $[x, y] = [x, [y, x]] = [x, x] = x$, again using B2 and B1. This shows (1). The proof of (2) is similar and left to the reader. $\square$

Lemma 3.7. *Let (X,φ) be a Smale space. Suppose that $x, y \in X$ satisfy $d(x,y) < \epsilon_X$ and also that*

$$d\big(x,[x,y]\big), d\big(y,[x,y]\big) < \epsilon_X.$$

Then

(1) $[x,y] \in X^s(x,\epsilon_X)$,

(2) $[x,y] \in X^u(y,\epsilon_X)$.

Proof. For (1), we have $[[x,y],x] = [x,x] = x$ by B3 followed by B1. Thus $[x,y] \in X^s(x,\epsilon_X)$. For (2), we have $[y,[x,y]] = [y,y] = y$ by B2 followed by B1. Thus $[x,y] \in X^u(y,\epsilon_X)$. □

Now we come to an important theorem, which shows that the existence of the bracket map really does capture the heuristic idea of having a system of local stable and unstable coordinates.

Theorem 3.8. *Let (X,φ) be a Smale space. There exists ϵ'_X satisfying $0 < \epsilon'_X \leq \epsilon_X/2$, such that, for every $\epsilon > 0$ with $0 < \epsilon \leq \epsilon'_X$, and every $x \in X$, the map*

$$[\,\cdot\,,\,\cdot\,] : X^u(x,\epsilon) \times X^s(x,\epsilon) \to X$$

is a homeomorphism onto its image, which is an open subset of X containing x.

Proof. Suppose that $y, z \in X$ satisfy

$$d(x,y) < \epsilon \leq \frac{\epsilon_X}{2} \text{ and } d(x,z) < \epsilon \leq \frac{\epsilon_X}{2}.$$

Then by the triangle inequality, $d(y,z) \leq \epsilon_X$. Thus the map is well defined.

Since $[\,\cdot\,,\,\cdot\,]$ is jointly continuous and satisfies $[x,x] = x$, there exists $0 < \delta \leq \epsilon_X$ such that, for every $x, y \in X$ with $d(x,y) \leq \delta$ we have that both

$$d\big(x,[x,y]\big) \leq \frac{\epsilon_X}{2} \text{ and } d\big(x,[y,x]\big) \leq \frac{\epsilon_X}{2}.$$

Choose $\epsilon'_X \leq \epsilon_X/2$ to be small enough that if $y, z \in X$ satisfy $d(x,y) \leq \epsilon'_X$ and $d(x,z)\epsilon'_X$, then $d(x,[y,z]) \leq \delta$. Then the map

$$h(y) := \big([y,x],[x,y]\big),$$

is defined on a neighbourhood of x and is evidently continuous. We claim that h is inverse to then map $[\,\cdot\,,\,\cdot\,] : X^u(x,\epsilon) \times X^s(x,\epsilon) \to X$. For $y \in X$ such that $d(x,y) < \epsilon'_X$, we have

$$[\,\cdot\,,\,\cdot\,] \circ h(y) = \big[\,[y,x],[x,y]\,\big] = [y,[x,y]\,] = [y,y] = y.$$

Now suppose that $y \in X^u(x,\epsilon)$ and $z \in X^s(x,\epsilon)$. Then

$$h([y,z]) = \big([\,[y,z],x],[x,[y,z]\,]\big) = \big([y,x],[x,z]\big) = [y,z].$$

This proves the claim. Finally, we show that the image of the map is open. Let $y \in X^u(x,\epsilon)$ and $z \in X^s(x,\epsilon)$. Since the bracket map is continuous, there is $\delta' > 0$ such that

$$h\big(B([y,z],\delta')\big) \subset B(x,\epsilon - d(x,y)) \times B(x,\epsilon - d(x,z)),$$

which is to say, $h(B([y,z],\delta'))$ is in the domain of $[\,\cdot\,,\,\cdot\,]$. Hence $B([y,z],\delta')$ is in the range of $[\,\cdot\,,\,\cdot\,]$, which proves the theorem. □

Proposition 3.9. *There exists a constant $\epsilon_1 > 0$ such that, for every $0 < \epsilon \leq \epsilon_1$, the following hold.*

(1) *For every $x, y \in X$, we have $d(\varphi^n(x),\varphi^n(y)) < \epsilon$ for every $n \geq 0$ if and only if $y \in X^s(x,\epsilon)$.*

(2) *For every $x, y \in X$, we have $d(\varphi^{-n}(x),\varphi^{-n}(y)) < \epsilon$ for every $n \geq 0$ if and only if $y \in X^u(x,\epsilon)$.*

Proof. There exists $0 < \epsilon_1 \leq \epsilon_X$ such that for every $x, y \in X$ with $d(x,y) < \epsilon_1$, then $d([y,x],x) < \epsilon_X$.

Let $0 < \epsilon \leq \epsilon_1$ and suppose that $y \in X^s(x,\epsilon)$. Then $d(\varphi(x),\varphi(y)) \leq \lambda_X d(x,y) < \epsilon$. Thus $\varphi(y) \in X^s(\varphi(x),\epsilon)$. By induction, we also have $d(\varphi^n(x),\varphi^n(x)) < \epsilon$ for every $n \geq 0$.

Conversely, suppose that $d(\varphi^n(x),\varphi^n(y)) < \epsilon$ for every $n \geq 0$. Then since $\epsilon \leq \epsilon_X$, the bracket $[\varphi^n(y),\varphi^n(x)]$ is defined for every $n \geq 0$. Since $[\varphi^n(y),\varphi^n(x)]$ is defined, we have $[\varphi^n(y),\varphi^n(x)] \in X^s(\varphi^n(x),\epsilon)$ for every $n \geq 0$. By B4 applied to φ^{-1} we have

$$\varphi^{-1}\big([\varphi^n(y),\varphi^n(x)]\big) = [\varphi^{n-1}(y),\varphi^{n-1}(x)]$$

and therefore

$$d\big(\varphi^{n-1}(x),[\varphi^{n-1}(y),\varphi^{n-1}(x)]\big) \leq \lambda_X d\big(\varphi^n(x),[\varphi^n(y),\varphi^n(x)]\big).$$

By induction, we see that

$$d\big(x,[y,x]\big) \leq \lambda^n d\big(\varphi^n(x),[\varphi^n(y),\varphi^n(x)]\big) \leq \lambda_X^n \epsilon_X.$$

Since $\lambda_X < 1$, we have $x = [y,x]$. Hence $y \in X^s(x,\epsilon)$.

The proof of (2) follows by applying (1) to the Smale space (X,φ^{-1}), see Remark 3.5. □

Observe that the previous proposition tells us that the bracket map only depends on $((X,d),\varphi)$. In other words, a dynamical system (X,φ) admits at most one Smale space structure.

Theorem 3.10. *With ϵ_1 as above, if $d(x,y) \leq \epsilon_X$ and $d(x,[x,y]), d(y,[x,y]) < \epsilon_1$, then*

$$\{[x,y]\} = X^s(x,\epsilon_1) \cap X^u(y,\epsilon_1).$$

Corollary 3.11. *If $x,y \in X$ and $d(\varphi^n(x),\varphi^n(y)) < \epsilon_1$ for every $n \in \mathbb{Z}$, then $x = y$.*

When a dynamical system (X,φ) satisfies the corollary above, we say that (X,φ) is *expansive* for the constant ϵ_1.

Although Smale spaces have a rich local structure, as with most dynamical systems, we are also interested with the global behaviour of the system.

Definition 3.12. Let (X,φ) be a Smale space. Two points $x,y \in X$ are called *stably equivalent*, denoted $x \sim_s y$ if

$$\lim_{n\to\infty} d(\varphi^n(x),\varphi^n(y)) = 0.$$

The *global stable set* of a point $x \in X$ is defined to be

$$X^s(x) := \big\{y \in X \mid\ x \sim_s y\big\}.$$

Similarly, to $x,y \in X$ are *unstably equivalent*, $x \sim_u y$ if

$$\lim_{n\to\infty} d(\varphi^{-n}(x),\varphi^{-n}(y)) = 0.$$

The *global unstable set* of $x \in X$ is

$$X^u(x) := \big\{y \in X \mid\ x \sim_u y\big\}.$$

Observe that both $\sim_s$ and $\sim_u$ are equivalence relations on X.

Clearly the local stable set of a point $x \in X$ is contained in the global stable set of X, and similarly for the local and global unstable sets. As we observed for the shifts of finite type in Example 3.1, the converse will not hold in general. However, we can characterize the global sets in terms of the local sets. This is useful for the purpose of putting a topology on these sets. The proof of the following proposition is straightforward.

Proposition 3.13. *Let (X,φ) be a Smale space and $0 < \epsilon \leq \epsilon_X$. Then*

(1) $X^s(x) = \bigcup_{n\geq 0} \varphi^{-n}(X^s(\varphi^n(x),\epsilon))$,

(2) $X^u(x) = \bigcup_{n\geq 0} \varphi^n(X^u(\varphi^{-n}(x,\epsilon)).$

We endow $X^s(x)$ and $X^u(x)$ with the inductive limit topology. With this topology, $X^s(x)$ and $X^u(x)$ are locally compact and Hausdorff.

Definition 3.14. Let (X,φ) be a Smale space. Two points $x,y \in X$ are *homoclinic*, written $x \sim_h y$, if $x \sim_s y$ and $x \sim_u y$. The *homoclinic set* of a point x is defined by

$$X^h(x) = \{y \in X \mid \; x \sim_h y\}.$$

Again, it is clear that $\sim_h$ is an equivalence relation on X. Observe that, unlike the stable and unstable relation, it doesn't make sense to talk about a "local" homoclinic set as, if $y \in X$ satisfies $[y,x] = x = [x,y]$ implies that the only element in such a set would be x itself.

We finish this section by collecting some facts about Smale spaces that will have important consequences for the C*-algebras we construct in Section 5. The next theorem is proved using the concept of *shadowing*, which we will not discuss here. For a discussion of shadowing in Smale spaces and a proof of the theorem, see [19].

Theorem 3.15. *Let (X,φ) be a Smale space. Then $\mathrm{Per}(X,\varphi)$ is a dense subset of $\mathrm{NW}(X,\varphi)$. In particular if (X,φ) is non-wandering, then $\mathrm{Per}(X,\varphi)$ is dense in X.*

Theorem 3.16. *Let (X,φ) be a Smale space. If (X,φ) is mixing, then, for every $x \in X$, the sets $X^s(x), X^u(x)$ and $X^h(x)$ are dense in X. Conversely, if (X,φ) is non-wandering and for every $x \in X$ the sets $X^s(x), X^u(x)$ and $X^h(x)$ are dense in X, then (X,φ) is mixing.*

Proof. Suppose that (X,φ) is mixing. To show that $X^s(x)$ and $X^u(x)$ are dense, it is enough to show that $X^s(x)$ is dense. Let $\delta > 0$ and let $y \in X$. We will show that $X^s(x) \cap B(y,\delta) \neq \varnothing$. Let $\epsilon > 0$ be sufficiently small so that $\epsilon < \epsilon_X$ and if $d(v,w) < \epsilon$ then $d([v,w],w) < \delta/2$. Let $(f^{n_j}(x))_{j\in\mathbb{N}}$ be a subsequence of $(f^n(x))_{n\in\mathbb{N}}$ which converges to a point $x_0 \in X$. Let $U = B(x,\delta/2)$ and $V = B(x_0,\epsilon/2)$. Since (X,φ) is mixing, there exists a positive integer N such that $f^n(U) \cap V \neq \varnothing$ for every $n \geq N$. Since $f^{n_j}(x)$ converges to x_0, there is a $n_j \geq N$ such that $f^{n_j}(x) \in X(x_0,\epsilon/2)$. Since $n_j \geq N$, there is $z \in U$ such that $f^{n_j}(z) \in V$. It follows that

$$w := f^{-n_j}\left([f^{n_j}(x), f^{n_j}(z)]\right),$$

is well defined and that $f^{n_j}(w) \in X^s(f^{n_j}(x),\delta/2)$. It follows that $w \in X^s(w)$.

To show that $X^h(x)$ is dense, we must show that $X^s(x) \cap X^u(x)$ is dense. In fact, we can prove something stronger: for any $x,y \in X$, $X^s(x) \cap X^u(y)$ is dense in X. (This will be useful when we define étale groupoids from the stable and unstable equivalence relations.) Let U be an open subset of X. Since the bracket map is continuous, there exists an open subset $V \subset U$ such that $d(v,w) < \epsilon_X$ for every $v,w \in V$ for which $[V,U] \subset U$. Since $X^s(x)$ and $X^u(x)$ are both dense, there exists $v \in X^s(x) \cap V$ and $w \in X^u(X) \cap V$. Then $[v,w]$ is a well-defined point which is in U and $X^s(x) \cap X^u(y)$. Thus $X^s(x) \cap X^u(y)$, and hence also $X^h(x)$, is dense.

Conversely, suppose that $X^s(x)$ and $X^u(x)$ are dense. Let U and V be non-empty open sets. Since (X,φ) is by assumption non-wandering, by the previous theorem there exists a periodic point $x \in U$. Let p be the period of x. Let $\epsilon > 0$ be sufficiently small so that $X^u(x,\epsilon) \subset U$. The sets $f^{kp}(X^u(x,\epsilon))$, $k \geq 1$ are increasing in diameter and their union is dense. The same is true for the sets $f^{kp}(X^u(f^j(x),\epsilon)$, $k \geq 1$, $1 \leq j < p$. Thus there exists $K \geq 1$ such that $f^{kp}(X^u(f^j(x),\epsilon)) \cap V$ is non-empty for every $k \geq K$ and every $0 \leq j < p$. Let $N = Kp$. For $n \geq N$, write $n = kp + j$ for some $k \geq K$ and $0 \leq j \leq p$. Then

$$f^n(U) \cap V \supset f^n(X^u(x,\epsilon)) \cap V = f^k p \circ f^j(X^u(x,\epsilon)) \cap V \supset F^k p(X^u(f^j(x),\epsilon)) \cap V \neq \varnothing,$$

which shows that (X,φ) is mixing. $\square$

Smale proved his decomposition theorem for Axiom A diffeomorphisms, see [28]. Ruelle defined Smale spaces as a way of axiomaitzing the behaviour of Axiom A diffeomorphisms, from which we have the following decomposition theorem, see [24].

Theorem 3.17 (Smale's decomposition theorem)**.** *Let (X,φ) be an irreducible Smale space. Then there exists $m \in \mathbb{Z}_{\geq 0}$ and subsets $X_1,\dots,X_{m-1}$ such that*

(1) $X = \bigsqcup_{i=1}^{m} X_i$,

(2) $\varphi(X_i) = X_{i+1 \mod m}$, $0 \leq i \leq m-1$,

(3) $(X_i, \varphi^m|_{X_i})$, $0 \leq i \leq m-1$, *is a mixing Smale space.*

We have the following consequence of Smale's decomposition theorem.

Proposition 3.18. *Let (X,φ) be an irreducible Smale space with decomposition into mixing components $X = \sqcup_{i=0}^{m-1} X_i$. Then*

(1) *$x \in X^s(y)$ if and only if there is $i \in \{0,\dots,m-1\}$ such that $x, y \in X_i$ and $x \in X_i^s(y)$,*

(2) *$x \in X^u(y)$ if and only if there is $i \in \{0,\dots,m-1\}$ such that $x, y \in X_i$ and $x \in X_i^u(y)$,*

(3) *$x \in X^h(y)$ if and only if there is $i \in \{0,\dots,m-1\}$ such that $x, y \in X_i$ and $x \in X_i^H(y)$.*

Thus

$$X^s(y) = \bigsqcup_{i=0}^{m-1} X_i^s(y), \quad X^u(y) = \bigsqcup_{i=0}^{m-1} X_i^u(y) \text{ and } X^h(y) = \bigsqcup_{i=0}^{m-1} X_i^h(y).$$

An important part of understanding a given topological system (X,φ) is understanding the set of φ-invariant Borel measures on X. A Borel measure μ on X is φ-invariant if $\mu(\varphi(B)) = \mu(B)$ for every Borel set $B \subset X$. In the case of a Smale space, we the following result.

Theorem 3.19 ([25, Theorem 1])**.** *Given an irreducible Smale space (X,φ), there exists a unique φ-invariant, entropy-maximizing probability measure μ_X on X which, for every $x \in X$ and $0 < \epsilon < \epsilon_X$, can be written locally as a product measure supported on $X^u(x,\epsilon) \times X^s(x,\epsilon)$.*

We call μ_X the *Bowen measure.* In the case of a shift of finite type it is the same as the *Parry measure*, see for example [16, Section 9.4].

The next theorem summarizes some of the key properties of the Bowen measure and its interaction with the bracket map.

Theorem 3.20 (see [14, Theeorem 1.1])**.** *Let (X,φ) be an irreducible Smale space. Let μ_X denote the Bowen measure of (X,φ) and, for any $x \in X$, let μ_x^s and μ_x^u denote the decomposition $\mu_X(A) = \mu_x^u \times \mu_x^s(A)$ for every $A \subset B(x,\epsilon_X)$. Let λ be defined by $h(\varphi) = \log(\lambda)$, where $h(\varphi)$ is the topological entropy of (X,φ).*

(1) *For every $x \in X$, μ_u^x and μ_x^s are non-finite regular Borel measures.*

(2) *Let $x \in X$ and $\epsilon > 0$. Then, for any pair of Borel sets $A \subset X^u(x,\epsilon)$ and $B \subset X^s(x,\epsilon)$ such that $[A,B]$ is defined, we have*
$$\mu_X([A,B]) = \mu_x^u(A)\mu_x^s(B).$$

(3) *For every $x, y \in X$, $\epsilon > 0$ and any Borel set $A \subset X^u(x,\epsilon)$ we have*
$$\mu_y^u([A,y]) = \mu_x^u(A),$$
whenever $d(x,y)$ and ϵ are sufficiently small so that $[A,y]$ is defined.

(4) *For every $x, y \in X$, $\epsilon > 0$ and Borel set $A \subset X^u(x,\epsilon)$ we have*
$$\mu_y^u([A,y]) = \mu_x^u(A),$$
whenever $d(x,y)$ and ϵ are sufficiently small so that $[A,y]$ is defined.

(5) *For every $x, y \in X$, $\epsilon > 0$ and Borel set $B \subset X^s(x,\epsilon)$ we have*

$$\mu_y^s([y,B]) = \mu_x^s(B),$$

whenever $d(x,y)$ and ϵ are sufficiently small so that $[y,B]$ is defined.

(6) *For every $x \in X$ we have*

$$\mu_{\varphi(x)}^s \circ \varphi = \lambda^{-1}\mu_x^s \text{ and } \mu_{\varphi(x)}^u \circ \varphi = \mu_x^u.$$

4 Étale groupoids

To construct a C*-algebra whose input data comes from a Smale space, we first construct étale groupoids from the three equivalence relations of Section 3. From this, we will construct a convolution *-algebra which can be completed into a C*-algebra. A good introduction to groupoids and their C*-algebras can be found in [21] or [27].

Definition 4.1. A *groupoid* consists of a set G, endowed with a unary operator

$$\cdot^{-1} : G \to G,$$

a distinguished subset $G^{(2)} \subset G \times G$ called the set of *composable pairs*, and a partially defined multiplication

$$G^{(2)} \to G, \quad (\gamma,\alpha) \mapsto \gamma\alpha,$$

satisfying the following compatibility conditions.

(1) For every $\gamma \in G$ we have $(\gamma^{-1})^{-1} = \gamma$,

(2) For every $\gamma, \alpha, \beta \in G$, if $(\gamma,\alpha),(\alpha,\beta) \in G^{(2)}$ then both $(\gamma,\alpha\beta)$ and $(\gamma\alpha,\beta) \in G^{(2)}$ and $\gamma(\alpha\beta) = (\gamma\alpha)\beta$.

(3) For every $\gamma \in G$, both (γ,γ^{-1}) and (γ^{-1},γ) are in $G^{(2)}$ and for every $(\gamma,\alpha) \in G^{(2)}$ we have $\gamma^{-1}\gamma\alpha = \alpha$ and $\gamma\alpha\alpha^{-1} = \gamma$.

Observe that $\gamma^{-1}\gamma$ behaves like a left unit for α whenever $(\gamma,\alpha) \in G^{(2)}$, and similarly, $\alpha\alpha^{-1}$ behaves like a right unit for g whenever $(\gamma,\alpha) \in \mathcal{G}^{(2)}$. For this reason, we refer to such elements as *units* and define the space of units to be

$$G^{(0)} = \{\gamma^{-1}\gamma \mid \gamma \in G\}.$$

We say that γ^{-1} is the inverse of γ

Given a groupoid G, we define the *range* and *source* maps, $r, s : G \to G^{(0)}$ respectively, by

$$r(\gamma) = \gamma\gamma^{-1}, \ \ \gamma \in G,$$

and

$$s(\gamma) = \gamma^{-1}\gamma, \ \ \gamma \in G.$$

It can be useful to think of elements of G as arrows between elements of $G^{(0)}$: an element $\gamma \in G$ defines an arrow from its source $\gamma^{-1}\gamma \in G^{(0)}$ to its range $\gamma\gamma^{-1} \in G^{(0)}$. Note that if $x \in G^{(0)}$ then we always have $r(x) = x = s(x)$. This can be made more precise by defining a groupoid to be a small category in which each morphism is an isomorphism. This is equivalent to Definition 4.1.

Example 4.2. Let G be a discrete group. Then the usual operations make G into a groupoid where $G^{(2)} = G \times G$, and $G^{(0)} = \{e\}$.

Just as for groups, inverses in a groupoid are unique. Indeed, if $\gamma \in G$ and $\eta \in G$ satisfies the properties of Definition 4.1 (3), then since $(\gamma, \eta) \in G^{(2)}$ so that

$$(\gamma^{-1}\gamma)\eta = \eta = \gamma^{-1}(\gamma\eta) = \gamma^{-1}.$$

Using the uniqueness of the inverse, a simple calculation allows us to deduce that if $(\gamma, \eta) \in G^{(2)}$ then $(\eta^{-1}, \gamma^{-1}) \in G^{(2)}$ and $\eta^{-1}\gamma^{-1}(\gamma\eta)^{-1}$. Thus

$$s(\gamma\eta) = \eta^{-1}\gamma^{-1}\gamma\eta = \eta^{-1}\eta = s(\eta)$$

while

$$r(\gamma\eta) = \gamma\eta\eta^{-1}\gamma^{-1} = \gamma^{-1}\gamma = r(\gamma).$$

Conversely, suppose that $\gamma, \eta \in G$ and $s(\gamma) = r(\eta)$. Then $(\gamma, \gamma^{-1}\gamma) \in G^{(2)}$ and $(\eta\eta^{-1}, \eta) \in G^{(2)}$ so $(\gamma, \eta\eta^{-1}\eta) = (\gamma, \eta) \in G^{(2)}$. Thus

$$G^{(2)} = \big\{(\gamma, \eta) \in G \mid \ s(\gamma) = r(\eta)\big\}.$$

Example 4.3. Let $\mathcal{R} \subset X \times X$ be an equivalence relation. For $(x, y) \in \mathcal{R}$. Define $(x, y)^{-1} = (y, x)$ and $G^{(2)} = \{((x, y), (z, w)) \in G^2 \mid \ y = z\}$ and let $(x, y)(y, z) = (x, z)$. Then $r(x, y) = (y, y)$ and $s(x, y) = (x, x)$, so the unit space $G^{(0)} = \{(x, x) \in X \times X\}$, which we identify with X.

For any $x \in X$ we define

$$G^x := \big\{\gamma \in G \mid \ r(\gamma) = x\big\}, \quad G_x := \big\{\gamma \in G \mid \ s(\gamma) = x\big\}, \quad G^x_x := G^x \cap G_x.$$

When G is a groupoid, then there is an equivalence relation on $G^{(0)}$ given by $\{(s(\gamma), r(\gamma)) \mid \ \gamma \in G\}$. The map $\gamma \mapsto (s(\gamma), r(\gamma))$ is a groupoid morphism which is an isomorphism exactly when G itself is an equivalence relation. In this case, we say that G is *principal*.

Definition 4.4. A *topological groupoid* is a groupoid G endowed with a locally compact topology which makes the unit space $G^{(0)}$ Hausdorff in the relative topology, and such that the maps

$$r, s, \cdot^{-1} : G \to G,$$

are continuous, and

$$G^{(2)} \to G, \ \ (g, h) \mapsto gh$$

is continuous with respect to the relative topology of the product topology on $G \times G$.

Definition 4.5. Let G be a topological groupoid. If $r, s : G \to G$ are local homeomorphisms, then we say that G is *étale*.

It is important to note that we ask $r, s : G \to G$ be continuous as maps into the groupoid G, not just its unit space $G^{(0)}$.

If G is a topological group which we view as a groupoid, then G is étale precisely when G is discrete. In this sense, the property of a topological groupoid being étale is analogous to a topological group being discrete.

Let G be a principal groupoid. We say that G is *minimal* when, for every $x \in G^{(0)}$, the set $\{y \in G^{(0)} \mid \ y \in r(s^{-1}(x))\}$ is dense in $G^{(0)}$. Thinking of G as an equivalence relation on $G^{(0)}$, this just means that the equivalence class of x is dense in $G^{(0)}$.

We say that a subset $U \subset G^{(0)}$ is *invariant* if $r(GU) \subset U$, where

$$GU := \big\{\gamma x \mid \ (\gamma, x) \in G^{(2)} \cap G \times U\big\}.$$

Since $(\gamma, x) \in G^{(2)}$ if and only if $s(\gamma) = r(x) = x$, and $s(\gamma x) = s(\gamma)$, we have that $GU = \{\gamma \in G \mid s(\gamma) \in U\} = s^{-1}(U)$.

Proposition 4.6. *Let G be a principal étale groupoid. The following are equivalent.*

(1) *G is minimal.*

(2) *If $E \subset G^{(0)}$ is a closed G-invariant subset, then $E \in \{\varnothing, G^{(0)}\}$.*

Proof. Suppose that $E \subset G^{(0)}$ is a non-empty proper closed G-invariant subset. Then for any $x \in W$, the orbit of x must be contained in E and hence is not dense. Thus G is not minimal. Conversely, if G is minimal and $E \subset G^{(0)}$ is a non-empty G-invariant subset, then there exists $x \in E$. But since $[x] \subset E$, E is closed and $[x]$ is dense, we must have $E = G^{(0)}$. □

4.1 Groupoid C*-algebras

The construction of C*-algebras from topological groupoids is due to Renault [21]. Here we give the construction for étale groupoids, although in fact as long as a topological groupoid admits a so-called *Haar system*, both the full and reduced C*-algebras can be formed. For étale groupoids, things are simplified because the Haar system is given by counting measures. We refer the reader to [21] for further details.

Let G be an étale groupoid. Let

$$C_c(G) := \big\{ f : G \to \mathbb{C} \mid \ f \text{ continuous with compact support} \big\}$$

be the set of continuous functions from G to $\mathbb{C}$ with compact support. We equip $C_c(G)$ with the structure of a *-algebra as follows. Addition is defined pointwise. Multiplication is given by the convolution product

$$fg(\gamma) = \sum_{\{(\gamma,\alpha)\in G^{(2)} \mid \ \gamma\alpha=\gamma\}} f(\gamma)g(\alpha), \quad f, g \in C_c(G), \gamma \in G,$$

and the involution is given by

$$f^*(\gamma) = \overline{f(\gamma^{-1})}, \quad f \in C_c(G), \gamma \in G.$$

That the sum in the definition of the convolution product is well defined follows from the fact that groupoid is étale, which implies that every compactly supported function in fact has finite support. Indeed, if $(\gamma, \alpha) \in G^{(2)}$ satisfy $\gamma\alpha = \gamma$, then $\gamma^{-1}\gamma\alpha = \alpha$, so $\alpha = \gamma^{-1}\gamma$. Thus the sum is over all $\gamma \in G$ such that $r(\gamma) = s(\gamma^{-1}) = r(\gamma)$. Since r is a local homeomorphism, it is in particular locally one-to-one. Since the support of f is compact, there are only finitely many γ with $r(\gamma) = r(\gamma)$ that are contained within the support of f.

For every $x \in G^{(0)}$, we define a *-homomorphism

$$\pi_x : C_c(G) \to \mathcal{B}(\ell^2(s^{-1}(x)))$$

by

$$\pi_x(f)(\xi)(\gamma) = \sum_{\{(\gamma,\alpha)\in G^{(2)} \mid \ \gamma\alpha=\gamma\}} f(\gamma)\xi(\alpha),$$

for $f \in C_c(G)$, $\xi \in \ell^2(s^{-1}(x))$, $\gamma \in s^{-1}(x)$.

For $f \in C_c(G)$, we define a C*-norm by

$$\|f\|_r := \sup_{x\in X} \|\pi_x(f)\|,$$

where the norm on the right hand side is the operator norm in $\mathcal{B}(\ell^2(s^{-1}(x)))$.

Definition 4.7. The *reduced groupoid* C*-*algebra* of an étale groupoid G, denoted $C^*_r(G)$, is the completion of $C_c(G)$ with respect to the norm $\| \cdot \|_r$.

One can also construct a *full* groupoid C*-algebra by completing $C_c(G)$ with respect to the norm given by the supremum of norms of all bounded *-representations of $C_c(G)$. We denote the full groupoid C*-algebra by $C^*(G)$. In general, $C^*(G)$ is "larger" in the sense that we have a surjective map $C^*(G) \to C^*_r(G)$. There can also be other completions of $C_c(G)$ whose lying in between the full and reduced groupoid C*-algebras.

5 C*-algebras of Smale spaces

Let $((X, d), \varphi)$ be an irreducible Smale space. Recall that two points $x, y \in X$ are

- stably equivalent, $x \sim_s y$, if $d(\varphi^n(x), \varphi^n(x)) \to 0$ as $n \to \infty$,

- unstably equivalent, $x \sim_u y$, if $d(\varphi^{-n}(x), \varphi^{-n}(y)) \to 0$ as $n \to \infty$,
- homoclinic, $x \sim_h y$, if $x \sim_s y$ and $x \sim_u y$.

Evidently, each of $\sim_s$, $\sim_u$, $\sim_h$ give us equivalence relations on X. Thus we can put groupoid structures on each of

$$G_s := \{(x,y) \mid\ x \sim_h y\}, \quad G_u := \{(x,y) \mid\ x \sim_u y\} \text{ and } G_h := \{(x,y) \mid\ x \sim_h y\},$$

which are respectively called the stable groupoid, unstable groupoid, and homoclinic groupoid of (X, φ). To construct C*-algebras, we would like to equip each of these with an étale topology. An obvious candidate is to equip each set with the relative topology of the product topology on $X \times X$. However, these will not in general be étale topologies. In particular, the equivalence classes in G_s and G_u will not even be countable. To rectify this, we restrict our equivalence classes to what is called an *abstract transversal*. This restricts the unit space X to a smaller subset that is more tractable, but which is "large" enough that it intersects each equivalence class so that we don't lose too much information. See [20] for more details.

Definition 5.1. Let (X, φ) be an irreducible Smale space. Let P and Q be a finite φ-invariant sets of periodic points of (X, φ). Define

$$G_s(P) := \big\{(x,y) \in X \times X \mid\ x \sim_s y \text{ and } x, y \in X^u(P)\big\}.$$

Likewise, if Q is a finite φ-invariant set of periodic points of (X, φ), then we define

$$G_u(Q) := \big\{(x,y) \in X \times X \mid\ x \sim_u y \text{ and } x, y \in X^s(Q)\big\}.$$

Note that the unit space of $G_s(P)$ is not X, but $X^u(P)$, which is no longer compact. Unlike the equivalence relation G_s the equivalence classes in $G_s(P)$ are countable.

Suppose that $x \sim_s y$. There exists $N \in \mathbb{N}$ such that

$$\varphi^N(y) \in X^s\Big(\varphi^N(x), \frac{\epsilon_X}{2}\Big) \text{ and } \varphi^N(x) \in X^s\Big(\varphi^n(y), \frac{\epsilon_X}{2}\Big).$$

Also, there is $0 < \delta \le \epsilon_X/2$ such that, for every $0 \le n \le N$ we have

$$\varphi^n(B(x,\delta)) \subset B\Big(\varphi^n(x), \frac{\epsilon_X}{2}\Big).$$

Shrinking δ if necessary, by the same argument, we have

$$\varphi^n(B(y,\delta)) \subset B\Big(\varphi^n(y), \frac{\epsilon_X}{2}\Big),$$

for every $0 \le n \le N$. Let $z \in X^u(x, \delta)$. Then

$$\big[\varphi^N(z), \varphi^N(y)\big] \in X^u(\varphi^N(y), \epsilon_X),$$

so

$$\varphi^{-N}\big(\big[\varphi^N(z), \varphi^N(y)\big]\big) \in X^u(y, \epsilon_X).$$

This gives us a map $h^u_x : X^u(x,\delta) \to X^u(y, \epsilon_X)$ defined by

$$h^u_x(z) = \varphi^{-N}\big(\big[\varphi^N(z), \varphi^N(y)\big]\big).$$

Similarly, there is a map $h^u_y : X^u(y,\delta) \to X^u(x, \epsilon_X)$ defined by

$$h^u_y(z) = \varphi^{-N}\big(\big[\varphi^N(z), \varphi^N(x)\big]\big).$$

If x and y are unstably equivalent, an analogous argument gives us a pair of maps $h^s_x : X^s(x,\delta) \to X^s(y, \epsilon_X)$ and $h^s_y : X^s(y,\delta) \to X^s(x, \epsilon_X)$ by

$$h^s_x(z) = \varphi^N\big(\big[\varphi^{-N}(z), \varphi^{-N}(y)\big]\big), \ \ z \in X^s(x,\delta),$$

and

$$h_y^s(z) = \varphi^N\big([\varphi^{-N}(z), \varphi^{-N}(x)]\big), \quad z \in X^s(y,\delta).$$

By [20], $h_x^u : X^u(x,\delta) \to X^u(y,\delta)$ is a local homeomorphism, mapping $X^u(x,\delta)$ homeomorphically to a neighbourhood of y, and hence a conjugating homeomorphism in the sense of Ruelle, see [23].

For such a 5-tuple v, w, δ, h_x^u, N as above, the sets

$$V(x,y,\delta,h_y^u,N) := \Big\{(h_y^u(z), z) \mid \ z \in X^u(y,\delta),\ h_y^u(z) \in X^u(x,\delta)\Big\} \subset G_s(P)$$

are basic sets generating a topology for $G_s(P)$.

Theorem 5.2 (see, for example, [13, Theorem 2.17])**.** *We have the following properties of $G_s(P)$ and the basic sets introduced in the previous paragraph.*

1. *The map h is a local homeomorphism;*
2. *$V(v,w,\delta,h,N)$ gives a neighbourhood base for a topology on $G_s(P)$;*
3. *$G_s(P)$ is an étale groupoid when we use this topology;*
4. *the unit space of $G_s(P)$ is $X^u(P)$ which is locally compact, but not compact.*

The analogous facts hold for $G_u(Q)$.

For G_h, the situation is less complicated, since the equivalence classes are already countable, without having to restrict to a subset of X. In this case, when $x \sim_h y$, for suitable δ and N we define mapes $h_x : B(x,\delta) \to B(y,\epsilon_X)$ and $h_y : B(y,\delta) \to B(x,\epsilon_X)$ by

$$h_x(z) = \big[h_x^u([z,x]), h_x^s([x,z])\big], \quad z \in B(X,\delta),$$

and

$$h_y(z) = \big[h_y^u([z,y]), h_y^s([x,y])\big], \quad z \in B(y,\delta).$$

For such $x \sim_h y$, N, δ, h_x and h_y, the subsets

$$V(x,y,h_y,\delta) = \big\{(h_y(z), z) \mid \ z \in B(y,\delta),\ h_y(z) \in B(x,\delta)\big\}$$

form a neighbourhood base for an étale topology on G_h.

Since $G_s(P)$ and $G_u(Q)$ are étale, we can constructed their reduced groupoid C*-algebras. One might be concerned that the choice of P and Q will result in different C*-algebras. Luckily, this is only the case up to *Morita equivalence*, which is a slightly weaker version of isomorphism for C*-algebras. Here it implies that for any choice of finite φ-invariant set of periodic points P, P' there is a *-isomorphism $\mathrm{C}^*_r(G_s(P)) \otimes \mathcal{K} \cong \mathrm{C}^*_r(G_s(P')) \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the C*-algebra of compact operators on a separable Hilbert space. Morita equivalent C*-algebras share many properties, including ideal structure, representation theory, K-theory, and nuclearity. In fact, we also have the following:

Theorem 5.3 ([18, cf. Theorem 3.1])**.** *Let (X,φ) be a mixing Smale space and P, Q finite φ-invariant subsets of periodic points. Then $\mathrm{C}^*_r(G_h)$ and $\mathrm{C}^*_r(G_s(P)) \otimes \mathrm{C}^*_r(G_u(Q))$ are Morita equivalent. In fact,*

$$\mathrm{C}^*(G_h) \otimes \mathcal{K} \cong \mathrm{C}^*_r(G_s(P)) \otimes \mathrm{C}^*_r(G_u(Q)).$$

Note that the theorem in [18] actually refers to the full groupoid C*-algebras, and the tensor product there is the maixmal tensor product. The fact that the above is true follow from the fact that each groupoid is *amenable*. The notion of amenability for a groupoid was introduced as a generalization of amenability for a topological group. The interested reader can find the definition and properties of an amenable groupoid in [1]. Most importantly here, if G is an amenable étale groupoid then its reduced and full groupoid C*-algebras coincide and are nuclear. One can also show the isomorphism directly for the reduced groupoid C*-algebras by representing them all on the same Hilbert space, see for example [13]. That the groupoids are amenable is a result of Putnam and Spielberg [20, Theorem 1.1].

Theorem 5.4. *The groupoids G_h, $G_s(P)$ and $G_u(Q)$ are amenable, hence the groupoid* C^**-algebras* $\mathrm{C}^*(G_h)$*,* $\mathrm{C}^*(G_s(P))$ *and* $\mathrm{C}^*(G_u(Q))$ *are nuclear.*

A C^*-algebra A is *simple* if the only closed, two-sided ideals in A are $\varnothing$ and A. When the C^*-algebra comes from an étale groupoids, we can deduce minimality from the structure of the groupoid.

For a groupoid G the *orbit* of a point $x \in G^{(0)}$ is the set

$$[x] = \{r(\gamma) \mid \gamma \in G,\ s(\gamma) = x\}.$$

Note that if G is an equivalence relation, the orbit of x is precisely the equivalence class of x. We can rephrase the notion of minimality given in the previous section to say that an étale groupoid $\mathcal{G}$ is minimal if $[x]$ is dense in $G^{(0)}$, for every $x \in G^{(0)}$.

Theorem 5.5. *Let G be a principal étale groupoid. If G is minimal, then* $\mathrm{C}^*_r(G)$ *is simple.*

This result, together with Theorem 3.16, implies the following:

Corollary 5.6. *Let (X, φ) be a Smale space, and let P, Q be finite φ-invariant subsets of* $\mathrm{Per}(X)$*. If (X, φ) is mixing, then* $\mathrm{C}^*(G_h)$*,* $\mathrm{C}^*(G_s(P))$ *and* $\mathrm{C}^*(G_u(Q))$ *are simple.*

Recall that Smale's decomposition theorem says that any irreducible Smale space decomposes into finitely many mixing components which are cylically permuted by the homeomorphism. This allows us to determine precisely the ideals of the C^*-algebra of the homoclinic groupoid of an irreducible Smale space.

Theorem 5.7. *Let (X, φ) be an irreducible Smale space with decomposition into mixing components given by $X = \bigsqcup_{i=0}^{m-1} X_i$. Let*

$$G_{i,H} := \{(x, y) \in X_i \times X_i \mid x \sim_h y\},$$

$0 \le i \le m-1$*. Then*

$$\mathrm{C}^*(G_h) \cong \mathrm{C}^*(G_{0,H}) \oplus \mathrm{C}^*(G_{1,H}) \oplus \cdots \oplus \mathrm{C}^*(G_{m-1,H}).$$

Now let us turn to other elements of a Smale space are reflected in the corresponding C^*-algebras.

Let (X, φ) be an irreducible Smale space. Let μ_X denote the Bowen measure of (X, φ) and, for any $x \in X$, let μ_x^s and μ_x^u denote the decomposition $\mu_X(A) = \mu_X^u \times \mu_x^s(A)$ for every $A \subset B(x, \epsilon_X)$.

The Bowen measure induces a trace on the C^*-algebras associated to (X, φ) as follows.

Let P and Q be finite, φ-invariant sets of periodic points. Define measures on the unit spaces $X^s(P)$ and $X^u(Q)$ of $G_s(P)$ and $G_u(Q)$, respectively, by

$$\mu^s := \sum_{x \in P} \mu_x^s \ \text{ and } \ \mu^u := \sum_{x \in Q} \mu_x^u.$$

For $f \in C_c(G_s(P))$, define

$$\tau^s(f) := \int\limits_{X^u(P)} f(x, x)\, d\mu^u,$$

and similarly, for $f \in C_c(G_u(Q))$, define

$$\tau^u(f) := \int\limits_{X^s(Q)} f(x, x)\, d\mu^s.$$

Since μ_X is φ-invariant, τ^s and τ^u define traces on $C_c(G_s)$ and $C_c(G_u)$. These can then be extended to faithful, semi-finite traces on $\mathrm{C}^*(G_s(P))$ and $\mathrm{C}^*(G_u(Q))$, respectively.

For the groupoid G_h, where the unit space is all of X, we can simply integrate a function in $C_c(G_h)$ with respect to μ_X, that is, we define

$$\tau^h(f) := \inf_X f(x, x)\, d\mu_X, \quad f \in C_c(G_h).$$

The map τ^h is bounded and hence extends to a tracial state on $\mathrm{C}^*(G_h)$. When (X, φ) is mixing, this is the unique tracial state on $\mathrm{C}^*(G_h)$ [12].

5.1 Classification of homoclinic C*-algebras of mixing Smale spaces

We saw in Corollary 5.6 that if (X, φ) is a mixing Smale space, then the associated groupoid C*-algebras are simple. Since they are C*-algebras of amenable groupoids, they moreover are in the UCT class [30]. Since X is a metric space, they are moreover separable and in the case of $\mathrm{C}^*(G_h)$, unital. Thus we would like to know if they are classifiable by the Elliott invariant, which requires showing that they have finite nuclear dimension.

We do so by considering a notion of dimension for the underlying groupoid: *dynamic asymptotic dimension.*

Definition 5.8 ([11, Definition 5.1])**.** We say that $\mathcal{G}$ has *dynamic asymptotic dimension at most d* if, for every open relatively compact $K \subset \mathcal{G}$, there exist open sets $U_0, \ldots, U_d \subset \mathcal{G}^{(0)}$ satisfying the following:

(1) $\{U_0, \ldots, U_d\}$ covers $s(K) \cup r(K)$;

(2) for every $i = 0, \ldots, d$, the groupoid generated by $\{g \in K \mid \ s(g), r(g) \in U_i\}$ is a relatively compact sub-groupoid of $\mathcal{H}$.

Theorem 5.9 ([11, Theorem 8.6])**.** *Let X be a locally compact metric space and $\mathcal{G} \subset X \times X$ an étale principal groupoid with dynamic asymptotic dimension at most d. Then*

$$\dim_{\mathrm{nuc}}(\mathrm{C}^*_r(\mathcal{G})) \leq (d+1)(\dim X + 1) - 1.$$

It follows from the definition of a Smale space (X, φ) together with [Mañé, 1979] that X must be finite dimensional.

Theorem 5.10 ([7, Theorem 3.7])**.** *Let (X, φ) be a mixing Smale space and P a set of periodic points with $\varphi(P) = P$. Then $G_s(P)$ and $G_u(P)$ have finite dynamic asymptotic dimension.*

The proof uses the fact that homeomorphism φ induces a groupoid automorphism $\alpha : G_s(P) \to G_s(P)$. By repeatedly applying the homeomorphism we can just as well look at $\alpha^n(K)$. By choosing large enough n we can assume that for every pair of points $x, y \in s(K) \cup r(K)$ we have that $d(\varphi^n(x), \varphi^n(y))$ is very small. Then, the groupoid starts to look like $X(P)$, and we arrive at the the estimate from the covering dimension of X.

Theorem 5.11 ([7, Corollary 3.8])**.** *Let (X, φ) be a mixing Smale space and P a set of periodic points with $\varphi(P) = P$. Then $\mathrm{C}^*(G_s(P))$, $\mathrm{C}^*(G_u(P))$ and $\mathrm{C}^*(G_h)$ all have finite nuclear dimension.*

Note that for $\mathrm{C}^*(G_h)$, the result is not proved using the dynamic asymptotic dimension. However it holds that

$$\mathrm{C}^*(G_h) \otimes \mathcal{K} \cong \mathrm{C}^*(G_s) \otimes C^*(G_u),$$

so the nuclear dimension of $\mathrm{C}^*(G_h)$ can be estimated from the nuclear dimension of $\mathrm{C}^*(G_s(P))$ and $\mathrm{C}^*(G_u(P))$.

Theorem 5.12. *Let (X, φ) and (Y, ψ) be mixing Smale spaces. Let A denote the homoclinic C*-algebra of (X, φ) and B the homoclinic C*-algebra of (Y, ψ). Suppose there is an isomorphism*

$$\psi : \mathrm{Ell}(A) \to \mathrm{Ell}(B).$$

*Then there is a *-isomorphism*

$$\Psi : A \to B,$$

which is unique up to approximate unitary equivalence and satisfies $\mathrm{Ell}(\Psi) = \psi$.

It turns out that $\mathrm{C}^*(G_s(P))$ and $\mathrm{C}^*(G_u(P))$ always contain notrivial projections [6]. In fact they have *real rank zero*, which implies a large supply of projections.

Let $p \in \mathrm{C}^*(G_s(P))$ be a nontrivial projection. Then $p\mathrm{C}^*(G_s(P))p$ is a unital C*-algebra which is Morita equivalent to $\mathrm{C}^*(G_s(P))$. Hence C*-algebras of the form $p\mathrm{C}^*(G_s(P))p$ are also classified by the Elliott invariant as in Theorem 5.12.

A Basics of C*-algebras

Definition A.1. A *-*algebra* is an algebra A (which in these notes, will always be over $\mathbb{C}$) equipped with a map $^* : A \to A$ called the *involution*, satisfying

$$(a^*)^* = a, \quad (ab)^* = b^*a^*,$$

for every $a, b \in A$.

Let A be a *-algebra. We say that an element $a \in A$ is

(1) *normal* if $a^*a = aa^*$,

(2) *self-adjoint* if $a^* = a$,

(3) a *projection* if $a^2 = a = a^*$.

If A is unital, we call an element $u \in A$ a *unitary* $u^*u = uu^* = 1_A$.

Definition A.2. An (abstract) C*-algebra is a *-algebra A which is complete with respect to a submultiplicative norm $\| \cdot \|$ satisfing the C*-equality:

$$\|a^*a\| = \|a\|^2 \text{ for every } a \in A.$$

A *Banach* *-algebra A is both a Banach algebra and a *-algebra with the compatibility condition

$$\|a^*\| = \|a\| \text{ for every } a \in A.$$

Note that for a unital Banach algebra A, we require that $\|1_A\| = 1$. For unital C*-algebras, this condition follows automatically from the C*-equality. We have the following strict inclusion of classes:

$$\{\text{C}^*\text{-algebras}\} \subsetneq \{\text{Banach } ^*\text{-algebras}\}.$$

For example, consider

$$\ell^1(\mathbb{Z}) = \Big\{(x_n)_{n\in\mathbb{Z}} \subset \mathbb{C}^{\mathbb{Z}} \mid \sum_{n\in\mathbb{Z}} |x_n| < \infty\Big\}$$

with norm $\|(x)_{n\in\mathbb{Z}}\| = \sum n \in \mathbb{Z}|x_n|$, involution $((x_n)_{n\in\mathbb{Z}})^* = (\overline{x_n})_{n\in\mathbb{Z}}$, pointwise addition, and multiplication defined by $(xy)_n = \sum_{j\in\mathbb{Z}} x_j y_{n-j}$. Then $\ell^1(\mathbb{Z})$ is a Banach *-algebra, but the norm does not satisfy the C*-equality. For example, let $x = (x_n)_{n\in\mathbb{Z}}$ satisfy $x_0 = 1$, $x_! = x_2 = -1$ and $x_n = 0$ otherwise. Then $\|x^*x\| = 5$ but $\|x\|^2 = 9$. Thus $\ell^1(\mathbb{Z})$ is not a C*-algebra.

Examples A.3. Some examples of C*-algebras are given by the following.

(1) Let $M_n := M_n(\mathbb{C})$ be the algebra of $n \times n$ matrices over $\mathbb{C}$. The involution sends a matrix to its adjoint, and the norm is the *operator norm*:

$$\|A\| := \sup\big\{\|Ax\|_{\mathbb{C}^n} \mid x \in \mathbb{C}^n, \ \|x\| \le 1\big\}.$$

(2) More generally, if H is any Hilbert space, then $\mathcal{B}(H)$ is a C*-algebra when equipped with the operator norm. In fact, any norm-closed subaglebra $A \subset \mathcal{B}(H)$ which is self-adjoint (that is, $A^* = A$) is a C*-algebra. Such a C*-algebra is said to be *concrete*.

(3) Let X be a locally compact Hausdorff space and

$$C_0(X) := \big\{f : X \to \mathbb{C} \mid f \text{ is continuous and vanishes at infinity}\big\}.$$

Then $C_0(X)$ is a C*-algebra with pointwise addition and multiplication, involution given by $f^*(x) = \overline{f(x)}$ for $f \in C_0(X)$, $x \in X$, and norm $\|f\| = \sup\limits_{x\in X} |f(x)|$. This is an example of a commutative C*-algebra.

A.1 Spectrum of an element

Let A be a unital Banach algebra and $a \in A$. The *spectrum* of a is the set

$$\mathrm{spec}\,(a) := \big\{\lambda \in \mathbb{C} \mid \ \lambda \cdot 1_A - a \text{ is not invertible}\big\},$$

and the *spectral radius* of a is the positive real number

$$r(a) := \sup_{\lambda \in \mathrm{spec}\,(a)} |\lambda|.$$

We always have that $r(a) \leq \|a\|$.

Let u be a unitary in a unital C*-algebra A. Then

$$\|1_a\| = \|u^*u\| = \|u\|^2,$$

so we must have that $\|u\| = 1$. It follows that if $\lambda \in \mathrm{spec}\,(a)$ it must satifsy $|\lambda| \leq 1$. On the other hand, since $u^*u = uu^* = 1_A$, we have that u is invertible with $u^{-1} = u^*$. It is easy to check that if $\lambda \in \mathrm{spec}\,(u)$ then $\lambda^{-1} \in \mathrm{spec}\,(u^*)$. Since u^* is also a unitary, we must have $|\lambda|^{-1}| \leq 1$, as well. Thus we see that

$$\mathrm{spec}\,(u) \subset \mathbb{T},$$

where $\mathbb{T}$ denotes the unit circle in the complex plane, $\mathbb{T} = \{\lambda \in \mathbb{C} \mid \ |\lambda| = 1\}$.

Proposition A.4. *Let* $a \in A_{\mathrm{sa}} := \{a \in A \mid \ a^* = a\}$. *Then* $\mathrm{spec}\,(a) \subset \mathbb{R}$.

Proof. In any unital Banach algebra we have

$$\Big\| \sum_{n=0}^{\infty} \frac{a^n}{n!} \Big\| \leq \sum_{n=0}^{\infty} \frac{\|a\|^n}{n!}\,,$$

which converges in $\mathbb{R}$. It follows that $\sum\limits_{n=0}^{\infty} \frac{a^n}{n!}$ converges in A to an element which we will denote e^a. One checks that $\varphi_a : \mathbb{R} \to A$, $t \mapsto e^{ta}$ is differentiable with derivative $a\varphi_a(t)$ and that $\varphi_a(0) = 1_A$. Moreover, φ_a is the unique function with these properties. In particular, if $a, b \in A$ and a and b commute, we must have that $e^{a+b} = e^a e^b$. It follows that e^a is always invertible with inverse $(e^a)^{-1} = e^{-a}$.

Now, if A is a unital C*-algebra and $a \in A_{sa}$, the e^{ia} is invertible with inverse $e^{-ia} = e^{(ia)^*} = (e^{ia})^*$. In other words, e^{ia} is a unitary. By our previous observation, $\mathrm{spec}\,(e^{ia}) \subset \mathbb{T}$. Let $\lambda \in \mathrm{spec}\,(a)$, and put

$$b := \sum_{n=2}^{\infty} \frac{i^n (a - \lambda \cdot 1_A)^{n-1}}{n!}.$$

Note that b commutes with $a - \lambda \cdot 1_A$. Since $a - \lambda \cdot 1_A$ is not invertible, neither is $e^{ia} - e^{i\lambda} \cdot 1_A$. Hence $e^{i\lambda} \in \mathrm{spec}\,(e^{ia}) \subset \mathbb{T}$, from which it follows that $\lambda \in \mathbb{R}$. □

We will make use of the following proposition. For a proof, see for example [17, Theorem 2.1.1].

Proposition A.5. *Let* $a \in A_{\mathrm{sa}}$. *Then* $r(a) = \|a\|$.

As a corollary, we see that, for a C*-algebra, the norm of an element is completely determined by spectral information, something is not the case for an arbitrary Banach algebra.

Corollary A.6. *If A is a* C*-*algebra with respect to the norm* $\|\cdot\|$, *then this is the unique norm* C*-*norm making A into a* C*-*algebra.*

Proof. Let $a \in A$. Then $a^*a \in A_{\mathrm{sa}}$, so

$$\|a\|^2 = \|a^*a\| = r(a^*a) = \sup_{\lambda \in \mathrm{spec}\,(a^*a)} |\lambda|.$$

Thus the norm of a depends only on its spectrum, meaning the norm is unique. □

In general, a Banach or C*-algebra need not have a unit. Examples include $C_0(X)$ when X is locally compact but not compact, or $\mathcal{K}$, the algebra of compact operators on a separable Hilbert space. If A is non-unital, we need to add a unit to make sense of the spectrum of an element.

Let A be a not-necessarily-unital algebra. Let $\widetilde{A} := A \oplus \mathbb{C}$ as a vector space, and define multiplication by

$$(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu), \quad a, b \in A, \quad \lambda, \mu \in \mathbb{C}.$$

Then $\widetilde{A}$ is a unital Banach algebra. If A is a *-algebra, we can define an involution on $\widetilde{A}$ by defining $(a, \lambda)^* = (a^*, \overline{\lambda})$. If A is moreover a Banach *-algebra with norm $\| \cdot \|_A$, we define a norm on $\widetilde{A}$ by

$$\|(a, \lambda)\| := \|a\|_A + |\lambda|, \quad a \in A, \quad \lambda \in \mathbb{C}.$$

However, if A is a C*-algebra, this will *not* make $\widetilde{A}$ into a C*-algebra. To define a C*-norm on $\widetilde{A}$, we view (a, λ) as a multiplication operator on A, that is,

$$(a, \lambda) : A \to A, \quad (a, \lambda)b \mapsto ab + \lambda b.$$

Then we equip $\widetilde{A}$ with the operator norm. In other words,

$$\|(a, \lambda)\| := \sup_{b \in A,\ \|b\| \leq 1} \|ab + \lambda b\|_A.$$

When A is a C*-algebra, we call $\widetilde{A}$ (with this norm) the *minimal unitization* of A, or for brevity, simply the *unitzation* of A. Note that $A \hookrightarrow \widetilde{A}$ is a *-perserving isometric algebra embedding, so if $a \in A$ we will write a instead of $(a, 0)$ when we consider it as an element in $\widetilde{A}$.

Let A be a non-unital C*-algebra and let $a \in A$. Then we define the spectrum of a to be the spectrum of $(a, 0) \in \widetilde{A}$, that is,

$$\operatorname{spec}(a) = \big\{\lambda \in \mathbb{C} \mid \ \lambda \cdot 1_{\widetilde{A}} - a \text{ is not invertible in } \widetilde{A}\big\}.$$

Let A and B be C*-algebras. A **-homomorphism* is an algebra map $\varphi : A \to B$ such that $\varphi(a^*) = \varphi(a)^*$ for every $a \in A$. Note that we do not require that φ is continuous – this will turn out to be automatic! If A and B are unital and $\varphi(1_A) = 1_B$, then we call φ a *unital *-homomorphism.* If B is unital and $\varphi : A \to B$ is a *-homomorphism, then φ has a to a unital *-homorphism $\widetilde{\varphi} : \widetilde{A} \to B$, that is, $\widetilde{\varphi}(a) = \varphi(a)$ for every $a \in A$ and $\widetilde{\varphi}(1_{\widetilde{A}}) = 1_B$.

Proposition A.7. *A *-homomorphism $\varphi : A \to B$ between two C*-algebras A and B is norm-decreasing. (In particular, φ is continuous.)*

Proof. Without loss of generality, we may assume that A, B, and φ are all unital. Then if $a \in A$ is invertible, $\varphi(a) \in B$ is also invertible. It follows that

$$\operatorname{spec}(\varphi(a)) \subset \operatorname{spec}(a),$$

for every $a \in A$. it follows from Corollary A.6 that $\|\varphi(a)\| \leq \|a\|$. $\square$

Corollary A.8. *A *-homomorphism is injective if and only if it is isometric. Every *-isomorphism is isometric.*

A.2 Spectrum of a C*-algebra

Let A be a Banach algebra. A *character* on A is an algebra homomorphism $\tau : A \to \mathbb{C}$. The *character space*, or *spectrum* of A is the set

$$\Omega(A) := \big\{\tau : A \to \mathbb{C} \mid \ \tau \text{ a character}\big\}.$$

By an *ideal* in a C*-algebra, we always mean a closed, self-adjoint, two-sided ideal, unless otherwise specified.

Proposition A.9. *Let A be a unital commutative* C*-*algebra. Then*

(i) $\tau(a) \in \operatorname{spec}(a)$ *for every* $a \in A$,

(ii) $\|\tau\| := \sup\{|\tau(b)| \mid \|b\|_A \le t\} = 1$,

(iii) $\Omega(A) \neq \varnothing$ *and if* $A \ncong \mathbb{C}$, *then* $\tau \mapsto \ker\tau$ *is a bijection between* $\Omega(A)$ *and the maximal ideals of* A.

Proof. Suppose $1_A \cdot \tau(a) - a$ is invertible with inverse b. Then

$$1 = \tau(1_A) = \tau\big(b(1_A \cdot \tau(a) - a)\big) = \tau(b)\tau(1_A \cdot \tau(a) - a) = \tau(b)(\tau(a) - \tau(a)) = \tau(b) \cdot 0 = 0,$$

a contradiction. Thus $1_A \cdot \tau(a) - a$ is not invertible, which is to say, $\tau(a) \in \operatorname{spec}(a)$, which shows (i).

For (ii), since $\tau(b) \in \operatorname{spec}(b)$, we have that $|\tau(b)| \le \|b\|$. Thus if $\|b\| \le 1$, we have $|\tau(b)| \le 1$. Since τ is an algebra homomorphism, $\tau(1_A) = 1$. Thus $\|\tau\| = 1$.

For (ii), suppose that A has a proper ideal. Then it is contained in a maximal ideal $J \subset A$. Since J is maximal, $A/J \cong \mathbb{C}$. Thus the quotient map $\pi : A \to A/J$ is a character with $\ker\tau = J$. If instead A has no proper ideals, then every element $a \in A \setminus \{0\}$ is invertible. Let a be a non-zero element. Then since $\operatorname{spec}(a)$ is non-empty and $0 \notin \operatorname{spec}(a)$ as a is invertible, we must have that $\lambda \in \operatorname{spec}(a)$ implies $\lambda \cdot 1_A - a = 0$. It follows that $A \cong \mathbb{C}$, and this isomorphism gives us a character. Thus $\Omega(A)$ is non-empty.

Conversely, suppose that $A \ncong \mathbb{C}$ and $\tau \in \Omega(A)$. Then $\ker(\tau)$ is easily seen to be a closed, self-adjoint, two-sided ideal which is moreover proper since $\tau(1_A) = 1$. Thus $\ker(\tau)$ is contained in a maximal ideal J. Then

$$J \cong A/J \subset A/\ker\tau \cong \mathbb{C}.$$

It follows that $\ker\tau = J$, hence is maximal. □

Theorem A.10. *Let A be a commutative Banach algebra. Then $\Omega(A)$ is locally compact and Hausdorff with respect to the weak∗-topology. If A is unital, then $\Omega(A)$ is compact.*

Proof. By Proposition A.9 (ii), we see that $\Omega(A) \setminus \{0\}$ is a weak∗-closed subset of the closed unit ball of A^*, the continuous linear dual of A. By the Banach–Alaoglu theorem, $\Omega(A) \setminus \{0\}$ is compact, whence $\Omega(A)$ is locally compact. If A is unital, 0 is not a character, so $\Omega(A)$ is compact. □

Let A be a commutative Banach algebra and let $a \in A$. Define

$$\widehat{a} : A^* \mapsto \mathbb{C}$$

by

$$\widehat{a}(\varphi) = \varphi(a).$$

Then $\widehat{A} \in C_0(\Omega(A)$. We call $\widehat{a}$ the *Gelfand transform* of a, and the map

$$\Gamma : A \to C_0(\Omega(A)), \quad a \mapsto \widehat{a}$$

is called the *Gelfand transform*.

Theorem A.11. *Let A be a commutative Banach algebra with non-empty character space. Then Γ is a norm-decreasing homomorphism and $r(a) = \|\widehat{a}\|$.*

Proof. It is straightforward to check that Γ is a homomorphism. We have

$$r(a) = \sup_{\lambda \in \operatorname{spec}(a)} |\lambda| = \sup_{\tau \in \Omega(A)} |\tau(a)| = \sup_{\tau \in \Omega(A)} |\widehat{a}(\tau)| = \|\widehat{a}\|.$$

Since $r(a) \le \|a\|$, we see that Γ is norm-decreasing. □

Theorem A.12. *Let A be a unital, commutative Banach algebra. Let $a \in A$ and let $B \subset A$ be the Banach subalgebra generated by a and 1_A. Then B is unital and commutative and*

$$\widehat{a} : \Omega(B) \to \operatorname{spec}(a)$$

is a homeomorphism.

If X is a locally compact Hausdorff space and $x \in X$, define

$$\operatorname{ev}_x : C_0(X) \to \mathbb{C}$$

by

$$\operatorname{ev}_x(f) = f(x).$$

It is easy to see that $\operatorname{ev}_x \in \Omega(C_0(X))$.

Theorem A.13. *Let X be a compact Hausdorff space. Then*

$$\Omega : X \to \Omega(C(X))$$

is a homeomorphism.

Proof. Let $(x_\lambda)_\Lambda \subset X$ be a net converging to $x \in X$. Then for every $f \in C(X)$,

$$\operatorname{ev}_{x_\lambda}(f) = f(x_\lambda) \to f(x) = \operatorname{ev}_x(f).$$

Thus $\operatorname{ev}_{x_\lambda} \to \operatorname{ev}_x$ in the weak$*$ topology, showing that the map Ω is continuous.

To show that Ω is injective, suppose that $x \neq y$. Then there exists $f \in C()$ such that $f(x) = 1$ and $f(y) = 0$. Hence $\operatorname{ev}_x(f) \neq \operatorname{ev}_y(f)$ and therefore $\operatorname{ev}_x \neq \operatorname{ev}_y$.

For surjectivity, let $\tau \in \Omega(C(X))$. We will show that $\ker \tau$ separates points. Suppose that $x \neq y$. Then as before, there exists $f \in C(X)$ such that $f(x) = 1$ and $f(y) = 0$. Observe that $f - \tau(f) \in \ker \tau$. Moreover,

$$(f - \tau(f))(x) \neq (f - \tau(f))(y).$$

By the Stone–Weierstrass theorem, there exists $z \in X$ such that $f(z) = 0$ for every $f \in \ker \tau$. Thus $f(z) = \tau(f)$ for every $f \in C(X)$. Thus $\tau = \operatorname{ev}_z$, and we have shown that Ω is surjective.

Since X is a compact Hausdorff space and Ω is a continuous bijection, it follows that Ω must be a homeomorphism. □

For what follows, we will use the observation: If A is a *-algebra and $a \in A$, then there exist $b, c \in A_{\mathrm{sa}}$ such that $a = b + ic$ and $bc = cb$. (We take $b = (a + a^*)/2$ and $c = (a - a^*)/2$).

Theorem A.14 (Gelfand–Naimark)**.** *Let A be a commutative* C**-algebra. Then the Gelfand transform*

$$\Gamma : A \to C_0(\Omega(A)), a \mapsto \widehat{a}$$

is a *-isomorphism (in particular, it is isometric).

Proof. Let $\tau \in \Omega(A)$. If $a \in A$ is self-adjoint, then $\operatorname{spec}(a) \subset \mathbb{R}$, so by Proposition A.9 (i), $\tau(a) \in \mathbb{R}$. If $a \in A$, let $b, c \in A_{\mathrm{sa}}$ satisfy $a = b + ic$. Then

$$\tau(a^*) = \tau(b^* - ic^*) = \tau(b - ic) = \tau(b) - i\tau(c) = \overline{\tau(b) + i\tau(c)} = \overline{\tau(a)}.$$

Thus $\Gamma(a^*) = \widehat{a}^* = (\widehat{a})^*$, from which it follows that Γ is a *-homomorphism. This implies that

$$\|\widehat{a}\|^2 = \|\widehat{a}^*\widehat{a}\| = \|\widehat{a^*a}\| = r(a^*a) = \|a^*a\| = \|a\|^2,$$

showing that Γ is isometric and hence injective. Surjectivity follows from the Stone–Weierstrass theorem: $\Gamma(A)$ separates points and contains function that do not simultaneously vanish on $\Omega(A)$. Thus $\Gamma(A) \cong C_0(\Omega(A))$. □

Theorem A.15. *Let A be a unital* C*-*algebra and $a \in A$ normal. Then*

$$\gamma : C(\mathrm{spec}\,(a)) \to A, \quad (z \mapsto z) \mapsto a$$

is an injective (hence isometric) *-*homomorphism and*

$$\gamma(C(\mathrm{spec}\,(a))) \cong \mathrm{C}^*(a, 1_a) \subset A,$$

where $\mathrm{C}^*(a, 1_a)$ *denotes the* C*-*subalgebra of A generated by a and 1_A.*

Proof. Since A is normal, $\mathrm{C}^*(a, 1_A)$ is commutative. Thus

$$\Gamma : \mathrm{C}^*(a, 1_A) \to C(\Omega(\mathrm{C}^*(a, 1_A)))$$

is a *-isomorphism. Since

$$h : \Omega(\mathrm{C}^*(a, 1_A)) \to \mathrm{spec}\,(a)$$

is a homeomorphism, the map

$$\psi : C(\mathrm{spec}\,(a)) \to C(\Omega(\mathrm{C}^*(a, 1_A)))$$

defined by

$$\psi(f) = f \circ h$$

is a *-isomorphism. Put $\gamma := \Gamma^{-1} \circ \psi$. Since $\mathrm{C}^*(a, 1_A)$ is generated by a and 1_A, we see that γ is the unique unital *-homomorphism satisfying $\gamma(f) = a$, where $f(z) = z$ for every $z \in \mathrm{spec}\,(a)$. Thus γ is an isometric *-homomorphism with

$$\gamma(C(\mathrm{spec}\,(a))) = \mathrm{C}^*(a, 1_A). \qquad \square$$

Let $p \in \mathbb{C}[z_1, z_2]$ be a polynomial in two commuting variables. Then if A is an algebra and $a, b \in A$ commute, $p(a, b)$ defines an element in A. In particular, if A is a C*-algebra and $a \in A^*$ is normal, $p(a, a^*) \in A$. Moreover,

$$\mathrm{spec}\,(p(a^*a)) = p(\mathrm{spec}\,(a), \mathrm{spec}\,(a^*)) = \left\{p(\lambda, \overline{\lambda}) \mid \ \lambda \in \mathrm{spec}\,(a)\right\}.$$

Since polynomials are dense in $C(\mathrm{spec}\,(a))$, using Theorem A.15, we can define, for any $f \in C(\mathrm{spec}\,(a))$ an element

$$f(a) = \gamma(f) \in \mathrm{C}^*(a, 1_A) \subset A.$$

This is called the *continuous functional calculus.*

The following is called the *spectral mapping theorem*:

Theorem A.16. *Let A be a* C*-*algebra and $a \in A$ normal. Then, for any $f \in C_0(\mathrm{spec}\,(a))$, the element $f(z) \in A$ is also normal and*

$$\mathrm{spec}\,(f(a)) = f(\mathrm{spec}\,(a)).$$

Furthermore, if $g \in C_0(\mathrm{spec}\,(f(a)))$, then

$$g(f(a)) = g \circ f(a).$$

Definition A.17. Let A be a C*-algebra. We say that $a \in A$ is *positive* if a is self-adjoint and $\mathrm{spec}\,(a) \subset [0, \infty)$.

For example, if $f \in C_0(X)$ where X is a locally compact Hausdorff space, then f is positive if and only if $f(x) \geq 0$ for every $x \in X$.

Positivity allows us to define a partial order on the self-adjoint elements in a C*-algebra A: Let $a, b \in A_{\mathrm{sa}}$. We define $\leq$ by putting $a \leq b$ if and only if $b - a$ is positive. Now, using the functional calculus we can prove things like the following.

- Every positive element has a unique positive square root: let $a \geq 0$ and identify $\mathrm{C}^*(a, 1_A)$ with $C(\operatorname{spec}(a))$, which maps a to the $f(z) = z$. Then define $\sqrt{a} := \sqrt{f}(a)$.

- If A is unital, it is spanned by its unitaries: Since A is spanned by the self-adjoint elements, it is enough to show that the norm one elements of A_{sa} is spanned by unitaries. If $a \in A_{\mathrm{sa}}$ and $\|a\| = 1$, then a^2 and $1-a^2$ are both positive. It follows that $\sqrt{1-a^2} \in A$. Let $u_1 = a - i\sqrt{1-a^2}$ and $u_2 = a + i\sqrt{1-a^2}$. Then u_1 and u_2 are unitaries and $a = u_1/2 + u_2/2$.

- If $a \in A_{\mathrm{sa}}$, then there are $b, c \geq 0$ such that $a = b - c$: let $f(x) = \max\{0, x\}$ and $g(x) = \max\{-x, 0\}$. Then $f(a), g(a) \geq 0$ and $a = f(a) - g(a)$. Note that we also have $f(a)g(a) = g(a)f(a) = 0$.

Theorem A.18. *For every $a \in A$, the element a^*a is positive.*

Proof. Suppose that $-a^*a$ is positive. Since $\operatorname{spec}(-a^*a) = \operatorname{spec}(-aa^*)$ we must also have that $-aa^*$ is positive. Write $a = b + ic$ with $a, b \in A_{\mathrm{sa}}$ and $ab = ba$. Then

$$a^*a = 2b^2 + 2c^2 - a^*a,$$

which is positive since $2b^2 + 2c^2$ and $-a^*a$ are both positive (that the sum of two positive commuting elements is again positive follows from the Gelfand–Naimark theorem). Thus $a^*a = 0$, thus $\|a^*a\| = 0$ hence $\|a\|^2 = 0$, so $a = 0$, in which case a is positive. Otherwise, $a \neq 0$. Write $a^*a = b - c$ for b, c positive and $bc = cb = 0$. Then

$$-(ac)^*(ac) = -ca^*ac = -c(b-c)c = c^3 \geq 0,$$

so $ac = 0$ by the above. Hence $c^2 = a^*ac = 0$. Thus $c = 0$ and $a^*a = b$ is positive. □

The notion of positivity is very important in the theory of C*-algebra. For a C*-algebra A, we denote the set of positive elements by A_+.

Let A and B be C*-algebras. A linear map $\varphi : A \to B$ is called *positive* if $\varphi(A_+) \subset B_+$. Note that any *-homomorphism is positive. If $B = \mathbb{C}$, and φ is positive, then we call φ a *positive linear functional*. A positive linear is called a *state* if $\|\varphi\| = 1$. (Here $\|\varphi\| = \sup\limits_{\|a\| \leq 1} |\varphi(a)|$.) A positive linear functional is called a *trace* if $\varphi(ab) = \varphi(ba)$ for every $a, b \in A$ and a *tracial state* if φ is both a state and trace. We denote the state space of A by $S(A)$, and the tracial state space of A by $T(A)$. A positive linear function $\varphi : A \to \mathbb{C}$ is *faithful* if $\varphi(a^*a) = 0$ implies that $a = 0$.

Theorem A.19. *Any positive linear functional is bounded.*

Proof. Suppose not. Then there is a sequence $(a_n)_{n\in\mathbb{N}}$ in A with $\|a_n\| \leq 1$ for every $n \in \mathbb{N}$ and such that $|\varphi(a_n)| \to \infty$ as $n \to \infty$. Since $\|a_n\| \leq 1$, we have that $\sum\limits_{n=2}^{\infty} a^n/2^n$ converges to some $b \in A$. But then

$$\varphi(b) = \varphi\Big(\sum_{n=2}^{\infty} \frac{a^n}{2^n}\Big) \geq \varphi\Big(\sum_{n=2}^{N} \frac{a^n}{2^n}\Big) > N \text{ for every } N \in \mathbb{N},$$

which is impossible. Thus φ must be bounded. □

Using this, one can show that positive linear functionals satisfy the Cauchy–Schwarz inequality:

$$|\varphi(a^*b)|^2 \leq \varphi(a^*a)\varphi(b^*b) \text{ for every } a, b \in A.$$

Definition A.20. Let A be a C*-algebra. An *approximate unit* for A is an increasing net $(u_\lambda)_\Lambda \subset A_+$ such that

$$\lim_\Lambda au_\lambda = \lim_\Lambda u_\lambda a = a,$$

for every $a \in A$.

Every C*-algebra contains an approximate unit.

Theorem A.21. *Let A be a* C*-*algebra and $(u_\lambda)_\Lambda$ an approximate unit. Then $\varphi \in A^*$ is positive if and only if $\lim_\Lambda \varphi(u_\lambda) = \|\varphi\|$. In particular, if A is unital then φ is positive if and only if $\varphi(1_A) = \|\varphi\|$.*

The idea of the proof is as follows. For one direction, assume that φ is positive. Then we use boundedness to show that $\lim_\Lambda \varphi(u_\lambda) = r \in \mathbb{R}_+$, $r \leq 1$ and thus $r \leq \|\varphi\|$. Then use the Cauchy–Schwarz inequality to show that $\|\varphi\|^2 \leq r\|\varphi\|$ so that $\|\varphi\| = r$. For the other direction, suppose that a is self-adjoint. Then we have $\varphi(a) \in \mathbb{R}$. If $a \in A_+$ with $\|a\| \leq 1$, then $u_\lambda - a$ is self-adjoint and $u_\lambda - a \leq u_\lambda$. Thus $\lim \varphi(u_\lambda - a) \leq \|\varphi\|$ so $\varphi(a) \geq 0$.

Corollary A.22. *If A is non-unital and $\varphi : A \to \mathbb{C}$ is a positive linear functional, then there is a unique positive linear functional $\widetilde{\varphi} : \widetilde{A} \to \mathbb{C}$, extending φ.*

In fact, with a little more work, the Hahn–Banach theorem gives us the following:

Proposition A.23. *Let A be a* C*-*algebra and $B \subset A$ a* C*-*subalgebra. Then every positive linear functional*

$$\varphi : B \to \mathbb{C}$$

admits an extension

$$\widetilde{\varphi} : A \to \mathbb{C}.$$

Proposition A.24. *Let A be a non-zero* C*-*algebra and $a \in A$ a normal element. The there exists $\varphi \in S(A)$ such that $\varphi(a) = \|a\|$.*

Proof. It follows from Corollary A.22 that we may assume that B and A are unital. Let $B \subset A$ be the C*-subalgebra generated by a and 1_A, $B = \mathrm{C}^*(a, 1_A)$. Since B is commutative, $\widehat{a} \in C(\Omega(B))$ such that $\varphi(a) = \widehat{a}(\varphi) = \|a\|$. Since $\varphi(1_A) = 1$, there exists a positive extension $\widetilde{\varphi} : A \to \mathbb{C}$ and $\varphi(a) = \|a\|$. ☐

A *-representation of a C*-algebra A is a pair (H, π), where H is a Hilbert space and

$$\pi : A \to \mathcal{B}(H)$$

is a *-homomorphism. When π is injective, we say that the representation is *faithful.*

Let $\varphi : A \to \mathbb{C}$ be a positive linear functional. Let

$$N_\varphi := \{a \in A \mid \varphi(a^*a) = 0\}.$$

Then N_φ is a closed left ideal in A (one can check this using the Cauchy–Schwarz inequality). Define

$$\langle \cdot, \cdot \rangle : A/N_\varphi \times A/N_\varphi \to \mathbb{C}$$

by

$$\langle a + N_\varphi, b + N_\varphi \rangle = \varphi(a^*b).$$

This defines an inner product on A/N_φ, allowing us to define a Hilbert space

$$H_\varphi := \overline{A/N_\varphi},$$

where the closure is respect to the norm induced by the inner product.

For $a \in A$, define

$$\pi_\varphi(a) : A/N_\varphi \to A/N_\varphi$$

by

$$\pi_\varphi(a)(b + N_\varphi) = ab + N_\varphi.$$

One can show that π_φ is bounded and hence extends to a bounded operator $\pi_\varphi : H_\varphi \to H_\varphi$. Define

$$\pi_\varphi : A \to \mathcal{B}(H_\varphi), \quad a \mapsto \pi_\varphi(a).$$

This is a *-homomorphism, hence (H_φ, π_φ) is a *-representation of A.

Definition A.25. Let A be a C*-algebra and $\varphi : A \to \mathbb{C}$ be a positive linear functional. The representation (H_φ, π_φ) is called the *Gelfand–Naimark–Segal* (GNS) representation of A associated to φ.

Given a family $(H_\lambda, \pi_\lambda)_{\lambda\in\Lambda}$ of representations, we define their direct sum by

$$\bigoplus_{\lambda\in\Lambda} \pi_\lambda : A \to \mathcal{B}\Big(\bigoplus_{\lambda\in\Lambda} H_\lambda\Big), \quad a \mapsto (\pi_\lambda(a_\lambda))_{\lambda\in\Lambda}.$$

The representation $\big(\bigoplus_{\varphi\in S(A)} H_\varphi, \bigoplus_{\varphi\in S(A)} \pi_\varphi\big)$ is called the *universal representation* of A.

Theorem A.26 (Gelfand–Naimark)**.** *Let A be a* C**-algebra. Then the universal representation of A is faithful.*

Proof. A direct sum representation is faithful if there exists $\lambda \in \Lambda$ such that $\pi_\lambda(a) \neq 0$. For every $a \in A$, there exists $\varphi \in A^*$ such that $\varphi(a) = \|a\|$. Thus $\pi_\varphi(a) \neq 0$. □

B Further structural properties

In the first Appendix, we developed the basic C*-algebra theory. In particular we saw that every commutative C*-algebra is, up to *-isomorphism, the algebra of functions on a locally compact Hausdorff space. We also saw that, up to *-isomorphism, every abstract C*-algebra is concrete, that is, it is a closed self-adjoint subalgbera of $\mathcal{B}(H)$ for some Hilbert space H. In this Appendix, we briefly mention further structural properties.

Let A be a C*-algebra. A *trace* $\tau : A \to \mathbb{C}$ is a positive linear functional satisfying $\tau(ab) = \tau(ba)$ for every $a, b \in A$. If $\|\tau\| = 1$ then we call τ *tracial state.* If A is a unital C*-algebra, the *tracial state space* is the set

$$T(A) := \big\{\tau : A \to \mathbb{C} \mid \ \tau \text{ a tracial state}\big\}.$$

The tracial state space is an important invariant for C*-algebras.

Example B.1.

(i) Let $A = M_n$. Then $T(A) = \{\mathrm{tr}\}$ where tr denotes the normalized trace of a matrix.

(ii) Let $A = C(X)$. Then any state is a tracial state since $C(X)$ is commutative.

Given a unital C*-algebra, the tracial state space can be equipped with the weak-$*$ topology. With respect to this topology, $T(A)$ is a Choquet simplex, see [26, Theorem 3.1.18].

Definition B.2. Let A and B be C*-algebras. A linear map $\varphi : A \to B$ is *positive* if $\varphi(E \cap A_+) \subset B_+$. For any map $\varphi : A \to B$ and $n \in \mathbb{N}$ we can define $\varphi^{(n)} : M_n(A) \to M_n(B)$ by applying φ entry-wise. If $\varphi^{(n)} : M_n(A) \to M_n(B)$ is positive for every $n \in \mathbb{N}$ then we say φ is *completely positive* (c.p.).

If φ is also contractive, then we say φ is *completely positive contractive* (c.p.c.), and if φ is unital, we say it is a *unital completely positive map* (u.c.p.).

Definition B.3. A C*-algebra A has the *completely positive approximation property* if, for every finite subset $\mathcal{F} \subset A$ and every $\epsilon > 0$ there exist a finite dimensional C*-algebra F and completely positive contraction $\psi : A \to F$ and $\varphi : F \to A$ such that $\|\varphi \circ \psi(a) - a\| < \epsilon$.

By results of Choi and Effros [5] as well as Kirchberg [15], a C*-algebra A has the completely positive approximation property if and only if A is *nuclear.* A C*-algebra A is nuclear if the algebraic tensor product of A with any other C*-algebra B has a unique C*-completion.

One can think of the completely positive approximation property as the noncommutative analogue of a topological space admitting partitions of unity. This leads to the question of whether one can refine the notion of the completely positive approximation property to a noncommutative analogue of covering dimension. This was the motivation behind the introduction of the *nuclear dimension* in [31].

Definition B.4. Let φ be a completely positive contractive map $\varphi : A \to B$. We say that φ is *order zero* if for any $a, b \in A_+$ with $ab = ba = 0$, we have $\varphi(a)\varphi(b) = 0$.

Definition B.5. Let A be a separable C*-algebra. We say that A has *nuclear dimension* d, written $\dim_{\mathrm{nuc}} A = d$, if d is the least integer satisfying the following: For every finite subset $\mathcal{F} \subset A$ and every $\epsilon > 0$ there are a finite-dimensional, C*-algebra with $d+1$ ideals, $F = F_0 \oplus \cdots \oplus F_d$, and completely positive maps $\psi : A \to F$ and $\varphi : F \to A$ such that ψ is contractive, $\varphi|_{F_n}$ are completely positive contractive order zero maps and

$$\|\varphi \circ \psi(a) - a\| < \epsilon \text{ for every } a \in \mathcal{F}.$$

If no such d exists, then we say $\dim_{\mathrm{nuc}} A = \infty$.

We are interested in simple, separable, unital, infinite dimensional C*-algebras which have finite nuclear dimension because they can be classified by the so-called *Elliott invariant.* The Elliott invariant of a C*-algebra A is given by

$$\mathrm{Ell}(A) := \big(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), \rho\big),$$

where $(K_0(A), K_0(A)_+, [1_A], K_1(A))$ is the (pointed, ordered) K-theory of A, $T(A)$ the tracial state simplex, and $\rho : K_0(A) \times T(A) \to \mathbb{R}$ a pairing map defined by $\rho([p] - [q], \tau) = \tau(p) - \tau(q)$, for τ the (non-normalized) inflation of a tracial state to a suitable matrix algebra over A. The K-theory of a C*-algebra A consists of two abelian groups, K_0 and K_1, the former is derived from an equivalence relation on projections in matrix algebras over A, while the later comes from equivalence classes of unitaries in matrix algebras over A. Since the self-adjoint elements of a C*-algebra are partially ordered, the K_0-group can also be partially ordered, and here $K_0(A)_+$ refers to the positive cone of the K_0-group. Finally $[1_A]$ refers to the K_0 class of the unit of A.

Theorem B.6 (see, for example, [2, 4, 8–10, 29])**.** *Let A and B be separable, unital, simple, infinite-dimensional* C**-algebras with finite nuclear dimension and which satisfy the UCT. Suppose there is an isomorphism*

$$\psi : \mathrm{Ell}(A) \to \mathrm{Ell}(B).$$

*Then there is a *-isomorphism*

$$\Psi : A \to B,$$

which is unique up to approximate unitary equivalence and satisfies $\mathrm{Ell}(\Psi) = \psi$.

The UCT refers to the *Universal Coefficient Theorem* of Rosenberg and Schochet [22]. Loosely speaking, it allows one to lift information from KK-theory to the level of K-theory. Every known nuclear algebra satisfies the UCT. In particular the UCT is satisfied for all C*-algebras in this note. However, it remains an important open problem to determine whether nuclearity implies the UCT.

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